18.06 Spring 2012 – Problem Set 3

This problem set is due Thursday, March 1st, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Without asking anyone for help, write down an accurate definition of what it means for a matrix to be in reduced row echelon form (RREF).

2. TRUE or FALSE? (No need for explanation):

   (a) Every upper-triangular matrix is in reduced row echelon form?

   (b) Every lower-triangular matrix is in reduced row echelon form?

   (c) Every permutation matrix is in reduced row echelon form?

   (d) The following matrix is in reduced row echelon form?

   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

   (e) The reduced row echelon form of A is unique?

   (f) The full solution set of $Ax = b$, where A is $m \times n$ and $b \in \mathbb{R}^m$, is always a vector subspace of $\mathbb{R}^n$?

   (g) The difference $\mathbf{a} = \mathbf{x}_1 - \mathbf{x}_2$, between any two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ to $A\mathbf{x} = \mathbf{b}$, is a vector that belongs to the null space $N(A)$? (Apply the rule $A(\mathbf{x} + \lambda\mathbf{y}) = A\mathbf{x} + \lambda A\mathbf{y}$ to $A(\mathbf{x}_1 - \mathbf{x}_2)$ to answer the question).

3. Do Problems 20 & 23 from Section 3.2.

4. Do Problem 35 from Section 3.2.

5. Do Problems 3 & 8 from Section 3.3.

6. Do Problems 17 & 28 from Section 3.3.

7. Do Problems 5 & 16 from Section 3.4.

8. Do Problems 24 & 33 from Section 3.4.

9. Do Problem 9 from Section 3.5.

   (See Problem 10 on next page!)

10. In this exercise, we try MATLAB's function `null(A)` for finding a basis (i.e. a minimal set of spanning vectors = a maximal set of independent vectors) for the null space of a matrix. We also try `rref(A)` for finding the reduced row echelon form.

```
B = [1  0  0  0;
     0  0  1  0;
     0  0  0  1;
     0  1  0  0];

C = [1  2  1 -2;
     0  0  1  5;
     0  0  0  0;
     0  0  0  0];

D = [1  2  0  1;
     0  2  2  1;
     0  0  3  3;
     1  0  0  4];
```

(a) Using `null()`, find a basis of each of $N(B), N(C)$ and $N(D)$ (the column vectors in the matrix MATLAB outputs are the basis vectors). Same for $N(BC)$ and $N(DC)$.

(b) Figure out whether $N(C)$ and $N(DC)$ are the same subspaces of $\mathbb{R}^4$, as follows: $\longrightarrow$ MATLAB can easily perform this, if we make use of the following two facts, for V and W subspaces of $\mathbb{R}^n$ with given collections of vectors used for spanning them, respectively $\mathbf{v}_1, \ldots, \mathbf{v}_k$ spanning V and $\mathbf{w}_1, \ldots, \mathbf{w}_l$ spanning W.

<u>**Fact 1**</u>: A vector $\mathbf{b} \in \mathbb{R}^4$ belongs to V *if and only if* the system $A\mathbf{x} = \mathbf{b}$ has at least one solution, where $A = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_k]$ is the matrix which as columns has a collection of vectors we use to span V.

Example (2×2): In MATLAB we create the augmented matrix $[A|\mathbf{b}]$ and use the command `rref`.

```
A = [1 2;
    -1 -2];
b = [1;
     1];

>> A_aug_b = [A b]

A_aug_b =
     1      2  |   1
    -1     -2  |   1
```

```
>> rref(A_aug)

ans =
     1     2   |  0
     0     0   |  1
```

(Note: The augmentation bars in the output will not show in MATLAB).

Notice the zero row that has a non-zero entry to the right of the bar: This system $A\mathbf{x} = \mathbf{b}$ has no solution. Hence, $\mathbf{b} = [1, 1]^T$ is not in the subspace spanned by the columns of A.

<u>Fact 2</u>: Two subspaces are the same, $V = W$, *if and only if:*

i. Vectors spanning V lie in W, that is $\mathbf{v}_1, \ldots, \mathbf{v}_k \in W$ (so $V \subseteq W$), and

ii. Vectors spanning W lie in V, that is $\mathbf{w}_1, \ldots, \mathbf{w}_k \in V$ (so $W \subseteq V$).

Example: Referring to the previous example, the subspace V spanned by the vectors $\mathbf{b}$ and $[0, 1]^T$ cannot be the same as the subspace W spanned by the columns of A (since we saw $\mathbf{b} \notin W$).

Now, for using Fact 1 & Fact 2 in MATLAB to determine if $N(C)$ and $N(DC)$ are in fact the same, you will need the ":" option:

```
>> A(:,2)  %Example: Gives you the 2nd column from matrix A
```

Then proceed as in the examples, checking each basis vector from one space for membership of the other space.

(c) Which property of the square matrix D explains the result of your comparison of $N(C)$ and $N(DC)$? State this as a general rule, and put a box around it. Apply your rule to explain why $N(DC)$ and $N(BC)$ are the same subspace.

(d) Is $N(CB)$ the same as $N(C)$? Either use the method from (b) again (you can do it all at once using `rref([null(CB) null(C)])`, if you carefully read off the result!), or simply try applying CB to the basis vectors you found for $N(C)$, and vice versa.