18.06 Spring 2012 – Problem Set 2

This problem set is due Thursday, February 23rd, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

1. Do Problems 7 & 9 from Section 2.6.

   2.6.7. Given
   $$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix} \text{ and } L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1},$$
   what three elimination matrices E_{21}, E_{31}, E_{32} put A into its upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by E_{32}^{-1}, E_{31}^{-1}, and E_{21}^{-1} to factor A into L times U.

   Solution.
   $$E_{32}E_{31}E_{21}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix}$$
   and this gives
   $$U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$
   Taking the inverses of the elimination matrices, and then putting them together gives:
   $$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} U.$$
   □

   2.6.9. Showing why $A = LU$ is not possible.

   Solution. The 2×2 case: Multiplying the two matrices on the right shows that we must have $d = 0$, which is not allowed.

   The 3×3 case: Again, multiply the two matrices on the right to get $d = 1, e = 1, g = 0, l = 1$. Then we need $f = 0$, which is not allowed. □

2. Do Problem 13 & 23 from Section 2.6.

   2.6.13.

Solution.

$$\begin{pmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{pmatrix}$$

this works when $a \neq 0$, $a \neq b$, $b \neq c$, $c \neq d$ to get four pivots. □

2.6.23

Solution. A_2 has the pivots 5 and 9, because elimination on A starts in the upper left corner, with elimination on A_2. □

3. Do Problem 6 from Section 2.7.

   The transpose of a block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ **is** $M^T = \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}$**. Test an example. For this matrix to be symmetric, we need** $A = A^T, D = D^T$**, and** $B = C^T$ **(and hence** $C = B^T$**).**

4. Do Problem 22 from Section 2.7.

   Find the $PA = LU$ **factorizations (and check them) for**

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix} \textbf{ and } \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Solution.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

5. Do Problem 38 from Section 2.7.

   If you take powers of a permutation matrix, why is some P^k **eventually equal to** I**? find a** 5 **by** 5 **permutation matrix** P **so that the smallest power to equal** I **is** P^6**.**

Solution. Since there are only finitely many permutation matrices (in fact, exactly $n!$ of them), there must be two powers P^a and P^b that are the same, with $a > b$. Then since P is invertible by pset 1, $P^{a-b} = I$. □

6. Do Problems 17 from Section 3.1.

   Solution to 3.1.17:

   (a) The zero matrix is not invertible. Therefore, the set of invertible matrices is not closed under multiplication by scalars, since multiplying anything by 0 gives the zero matrix. Therefore it is not a subspace of $\mathbf{M}$.

   (b) The matrices $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$ are both clearly singular, but their sum is the identity matrix, which is obviously invertible. Thus the set of singular matrices is not closed under addition and so is not a subspace of $\mathbf{M}$.

7. Do Problem 23 from Section 3.1.

   Solution to 3.1.23:
   If we add an extra column $\mathbf{b}$ to a matrix A, then the column space gets larger unless $\mathbf{b}$ was already in the column space. If

$$A = \begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

   then the column space gets larger. If

$$A = \begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

   then it does not. The column space of A is the same as that of $[A\ \mathbf{b}]$ precisely when $\mathbf{b}$ can be written as a linear combination of the columns of A, i.e. when there exists a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$.

8. Do Problems 30 & 31 from Section 3.1.

   Solution to 3.1.30:

   (a) Suppose $\mathbf{a}$ is in $\mathbf{S}+\mathbf{T}$; then by definition there exist $\mathbf{s} \in \mathbf{S}$ and $\mathbf{t} \in \mathbf{T}$ such that $\mathbf{a} = \mathbf{s}+\mathbf{t}$. For λ a scalar we have $\lambda\mathbf{a} = \lambda(\mathbf{s}+\mathbf{t}) = \lambda\mathbf{s}+\lambda\mathbf{t}$. Since $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{V}$, the vector $\lambda\mathbf{s}$ is in $\mathbf{S}$ and the vector $\lambda\mathbf{t}$ is in $\mathbf{T}$. Thus we have written $\lambda\mathbf{a}$ as a sum of a vector in $\mathbf{S}$ and a vector in $\mathbf{T}$, which by definition means that $\lambda\mathbf{a} \in \mathbf{S}+\mathbf{T}$. This proves that $\mathbf{S}+\mathbf{T}$ is closed under multiplication by scalars.
   Now suppose $\mathbf{a}$ and $\mathbf{a}'$ are in $\mathbf{S}+\mathbf{T}$; by definition there exist $\mathbf{s}, \mathbf{s}' \in \mathbf{S}$ and $\mathbf{t}, \mathbf{t}' \in \mathbf{T}$ such that $\mathbf{a} = \mathbf{s}+\mathbf{t}$ and $\mathbf{a}' = \mathbf{s}'+\mathbf{t}'$. Then since $\mathbf{V}$ is a vector space we have $\mathbf{a}+\mathbf{a}' = (\mathbf{s}+\mathbf{t})+(\mathbf{s}'+\mathbf{t}') = (\mathbf{s}+\mathbf{s}')+(\mathbf{t}+\mathbf{t}')$. As $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{V}$, the vector $\mathbf{s}+\mathbf{s}'$ is in $\mathbf{S}$ and the vector $\mathbf{t}+\mathbf{t}'$ is in $\mathbf{T}$. Thus we have written $\mathbf{a}+\mathbf{a}'$ as a sum of a vector in $\mathbf{S}$ and a vector in $\mathbf{T}$, which by definition means that $\mathbf{a}+\mathbf{a}' \in \mathbf{S}+\mathbf{T}$. This proves that $\mathbf{S}+\mathbf{T}$ is closed under addition.

   (b) $\mathbf{S}\cup\mathbf{T}$ is the set of vectors that lie in either $\mathbf{S}$ or $\mathbf{T}$, whereas $\mathbf{S}+\mathbf{T}$ is the set of sums of vectors in $\mathbf{S}$ and $\mathbf{T}$. These are clearly not the same — for example, $\mathbf{S}\cup\mathbf{T}$ will not be a vector space unless $\mathbf{S}$ and $\mathbf{T}$ are the *same* line through the origin, while we proved above that $\mathbf{S}+\mathbf{T}$ *is* a vector space. The *span* of a subset of a vector space $\mathbf{V}$ is the set of vectors that can be written as linear combinations of elements of the set; since $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbb{R}^m$, the span of $\mathbf{S}\cup\mathbf{T}$ is $\mathbf{S}+\mathbf{T}$.

Solution to 3.1.31:
The space $\mathbf{C}(A)+\mathbf{C}(B)$ consists of those vectors in $\mathbb{R}^m$ that can be written as a sum of a linear combination of the columns of A and a linear combination of the columns of B. This is the same thing as the vectors that are linear combinations of the columns of A together with the columns of B, so we can take $M = [A\ B]$.

9. This problem is about the vector space of matrices for a fixed number of rows and columns.

   (a) Explain carefully why the set of all 7×11 matrices forms a vector space (What is $c\,A + d\,B$? Which matrix is the zero vector?). Describe the simplest list of matrices you can think of which, allowing arbitrary linear combinations, will yield *all* 7×11 matrices. There should be 77 different matrices in your answer.

   (b) How many real number-valued parameters would you use to (unambiguously) describe the vector space $S_{3\times3}$ of 3×3 symmetric matrices (e.g. the set of all 3×3 matrices A such that $A^T = A$)? Identify *all* vector subspaces of $S_{3\times3}$ (it may be convenient to refer to the parameters you've introduced).

   (c) The 2×2 matrices with equal row sums ($a+b$ and $c+d$ are the same number), and equal column sums ($a+c$ and $b+d$), is a vector space. Find two matrices so that all these matrices are linear combinations of those two.

Solution:

(a) $cA+dB$ is the matrix whose (i,j)-component is $cA_{ij}+dB_{ij}$. Since scalar multiplication and addition are defined component-wise, the associativity and commutativity of addition of matrices, as well as the associativity of scalar multiplication and its distributivity over addition, all follow from the same properties of $\mathbb{R}$. The zero matrix is the zero vector in this vector space. Let E^{ij} be the matrix with components

$$E^{ij}_{kl} = \begin{cases} 1 & k = i, l = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then any 7×11 matrix can be written as a linear combination of these 77 matrices:

$$A = \sum_{i=1}^{7} \sum_{j=1}^{11} A_{ij} E^{ij}.$$

Moreover, no subset of these matrices spans the set 7×11 matrices, since only those matrices A such that $A_{kl} = 0$ can be written as a linear combination of E^{ij}'s not including E^{kl}.

(b) A symmetric 3×3-matrix A is uniquely determined by its 6 upper-triangular components A_{ij} with $j \geq i$. A subspace of $S_{3\times3}$ is determined by some (independent) linear equations in these parameters.

(c) Consider $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$; these clearly have equal row sums and equal column sums. Suppose $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $a+b = c+d$ and $a+c = b+d$. Then

$a = -b + c + d$ and $a = b - c + d$; adding these we get $2a = 2d$, so $a = d$. Then $a + b = c + a$ so $b = c$ also. Thus

$$A = \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right) = aI + bJ.$$

10. The MATLAB command `A = double(rand(2,2) < 0.5)` gives a random 2×2 matrix where each entry is either 0 or 1 (with equal probabilities).

    (a) Make a plot of the distribution of the number of pivots of the row-reduced versions (in MATLAB, the command `rank(A)` gives this number) of these random matrices. Here's some sample code that you can copy-paste into MATLAB:

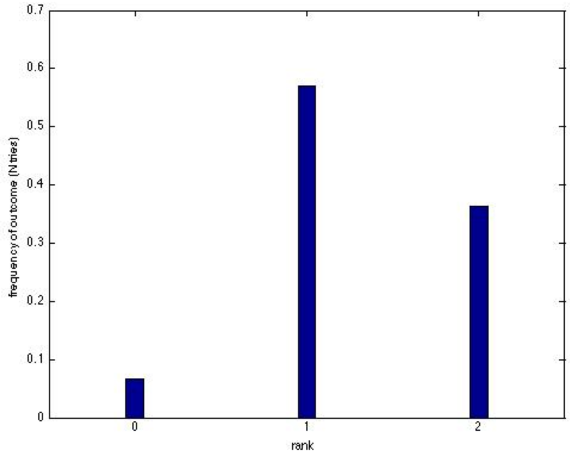
```
clear; N=1000; num_zeros=0; num_ones=0; num_twos=0;
for i = 1:N
 A = double(rand(2,2) < 0.5);
 if rank(A)==2
  num_twos = num_twos + 1; %Then add one to that counter!
 end
 if rank(A)==1
  num_ones = num_ones + 1;
 end
 if rank(A)==0
  num_zeros = num_zeros + 1;
 end
end
distrib = [num_zeros num_ones num_twos]/N
bar([0 1 2], distrib, 0.1)
```

    (b) Compare this to the exact probabilities of each value for the pivot number. Compute these by writing down all 16 possibilities and counting pivots.

    (c) Extend the code in (a) to work for 5×5 matrices, and again show a histogram plot.

    (d) For the 2×2 examples, what do you think the probability of having 2 pivots would be, if we took each matrix entry distributed continuously (and uniformly) in the *interval* $[0, 1]$? (No need to compute - but explain why!)

Solution:



(a)

(b) Only the zero matrix has no pivots, so the probability of 0 pivots is 1/16. The matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

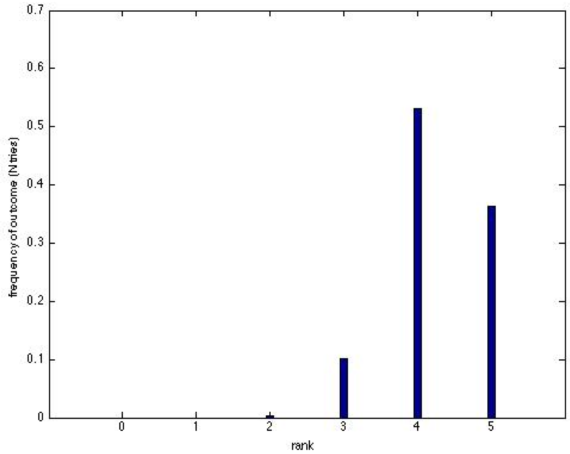
$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

have one pivot, so the probability of this is 9/16. The matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

have two pivots, so the probability of this is $6/16 = 3/8$.

(c)
```
clear; N=1000;
matsize=5;
nums=zeros(1, matsize+1);
for i = 1:N
 A=double(rand(matsize,matsize)<0.5);
 index = rank(A) + 1;
 nums(index) = nums(index)+1;
end
distrib = nums/N;
bar([0:1:matsize], distrib, 0.1)
xlabel('rank')
ylabel('frequency of outcome (N tries)')
```



(d) We can think of the entries of our matrices as points in the "hypercube" $[0,1] \times [0,1] \times [0,1] \times [0,1] \subseteq \mathbf{R}^4$. Since the entries are distributed uniformly, the probability that a matrix picked at random lies in some region in this subset of $\mathbf{R}^4$ equals the " 4-dimensional volume" of this region. A matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has less than two pivots precisely when it is singular, i.e. when its entries satisfy the equation $ad - bc = 0$. But the space where this equation holds is 3-dimensional, so its 4-dimensional volume is 0 (just like a curve in the plane has no area, or a surface in space has no volume). Thus a random matrix has 2 pivots with probability 1.