18.06 Professor Strang Final Exam May 23rd, 2012

Grading

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Your PRINTED name is: ____________________

Please circle your recitation:

r01	T 11	4-159	Ailsa Keating	ailsa
r02	T 11	36-153	Rune Haugseng	haugseng
r03	T 12	4-159	Jennifer Park	jmypark
r04	T 12	36-153	Rune Haugseng	haugseng
r05	T 1	4-153	Dimiter Ostrev	ostrev
r06	T 1	4-159	Uhi Rinn Suh	ursuh
r07	T 1	66-144	Ailsa Keating	ailsa
r08	T 2	66-144	Niels Martin Moller	moller
r09	T 2	4-153	Dimiter Ostrev	ostrev
r10	ESG		Gabrielle Stoy	gstoy

1 (12 pts.)

(a) - Find the eigenvalues and eigenvectors of A.

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{bmatrix}$$

(b) - Write the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of eigenvectors of A.

- Find the vector $A^{10}\mathbf{v}$.

(c) If you solve $\frac{d\mathbf{u}}{dt} = -A\mathbf{u}$ (notice the minus sign), with $\mathbf{u}(0)$ a given vector, then as $t \to \infty$ the solution $\mathbf{u}(t)$ will always approach a multiple of a certain vector $\mathbf{w}$.

- Find this steady-state vector $\mathbf{w}$.

2 (12 pts.)

Suppose A has rank 1, and B has rank 2 (A and B are both 3×3 matrices).

(a) - What are the possible ranks of $A + B$?

(b) - Give an example of each possibility you had in (a).

(c) - What are the possible ranks of AB?

- Give an example of each possibility.

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3 (12 pts.)

(a) - Find the three pivots and the determinant of A.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

(b) - The rank of $A - I$ is ______, so that $\lambda =$______ is an eigenvalue.

- The remaining two eigenvalues of A are $\lambda =$____________.

- These eigenvalues are all __________________, because $A^T = A$.

(c) The unit eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ will be orthonormal.

- Prove that:

$$A = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T + \lambda_3 \mathbf{x}_3 \mathbf{x}_3^T.$$

You may compute the $\mathbf{x}_i$'s and use numbers. Or, without numbers, you may show that the right side has the correct eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

4 (12 pts.)

This problem is about $x + 2y + 2z = 0$, which is the equation of a plane through $\mathbf{0}$ in $\mathbb{R}^3$.

(a) - That plane is the nullspace of what matrix A?

$A =$

- Find an orthonormal basis for that nullspace (that plane).

(b) That plane is the column space of many matrices B.

- Give two examples of B.

(c) - How would you compute the projection matrix P onto that plane? (A formula is enough)

- What is the rank of P?

5 (12 pts.)

Suppose $\mathbf{v}$ is any unit vector in $\mathbb{R}^3$. This question is about the matrix H.

$$H = I - 2\mathbf{v}\mathbf{v}^T.$$

(a) - Multiply H times H to show that $H^2 = I$.

(b) - Show that H passes the tests for being a symmetric matrix <u>and</u> an orthogonal matrix.

(c) - What are the eigenvalues of H?

You have enough information to answer for any unit vector $\mathbf{v}$, but you can choose one $\mathbf{v}$ and compute the λ's.

6 (12 pts.)

(a) - Find the closest straight line $y = Ct + D$ to the 5 points:

$$(t, y) = (-2, 0),\ \ (-1, 0),\ \ (0, 1),\ \ (1, 1),\ \ (2, 1).$$

(b) - The word "closest" means that you minimized which quantity to find your line?

(c) - If $A^T A$ is invertible, what do you know about its eigenvalues and eigenvectors? (Technical point: Assume that the eigenvalues are distinct – no eigenvalues are repeated).

7 (12 pts.)

This symmetric Hadamard matrix has orthogonal columns:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad \text{and} \quad H^2 = 4I.$$

(a) What is the determinant of H?

(b) What are the eigenvalues of H? (Use $H^2 = 4I$ and the trace of H).

(c) What are the singular values of H?

8 (16 pts.)

In this TRUE/FALSE problem, you should *circle* your answer to each question.

(a) Suppose you have 101 vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{101} \in \mathbb{R}^{100}$.

- Each v_i is a combination of the other 100 vectors: TRUE – FALSE

- Three of the v_i's are in the same 2-dimensional plane: TRUE – FALSE

(b) Suppose a matrix A has repeated eigenvalues $7, 7, 7$, so $\det(A - \lambda I) = (7 - \lambda)^3$.

- Then A certainly cannot be diagonalized ($A = S\Lambda S^{-1}$): TRUE – FALSE

- The Jordan form of A must be $\mathcal{J} = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{bmatrix}$: TRUE – FALSE

(c) Suppose A and B are 3×5.

- Then $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$: TRUE – FALSE

(d) Suppose A and B are 4×4.

- Then $\det(A + B) \leq \det(A) + \det(B)$: TRUE – FALSE

(e) Suppose $\mathbf{u}$ and $\mathbf{v}$ are orthonormal, and call the vector $\mathbf{b} = 3\mathbf{u} + \mathbf{v}$. Take V to be the line of all multiples of $\mathbf{u} + \mathbf{v}$.

- The orthogonal projection of $\mathbf{b}$ onto V is $2\mathbf{u} + 2\mathbf{v}$: TRUE – FALSE

(f) Consider the transformation $T(x) = \int_{-x}^{x} f(t)dt$, for a fixed function f. The input is x, the output is $T(x)$.

- Then T is always a linear transformation: TRUE – FALSE

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This is the end of 18.06. Hope you enjoyed learning Linear Algebra!