18.06 Problem Set 10 Solution

Due Thursday, 29 April 2009 at 4 pm in 2-106.
Total: 100 points

Section 6.6. Problem 12. These Jordan matrices have eigenvalues $0, 0, 0, 0$. They have two eigenvectors (one from each block). But the block sizes don't match and they are *not similar*:

$$J = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \text{and} \qquad K = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

For any matrix M, compare JM with MK. If they are equal show that M is not invertible. Then $M^{-1}JM = K$ is impossible; *J is not similar to K.*

Solution (4 points) Let $M = (m_{ij})$. Then

$$JM = \begin{pmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad MK = \begin{pmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{pmatrix}.$$

If $JM = MK$ then

$$m_{21} = m_{22} = m_{24} = m_{41} = m_{42} = m_{44} = 0,$$

which in particular means that the second row is either a multiple of the fourth row, or the fourth row is all 0's. In either of these cases M is not invertible.

Suppose that J were similar to K. Then there would be some invertible matrix M such that $K = M^{-1}JM$, which would mean that $MK = JM$. But we just showed that in this case M is never invertible! Contradiction. Thus J is not similar to K.

Section 6.6. Problem 14. Prove that A^T is *always similar* to A (we know that the λ's are the same):

1. For one Jordan block J_i: find M_i so that $M_i^{-1}J_iM_i = J_i^T$ (see example 3).
2. For any J with blocks J_i: build M_0 from blocks so that $M_0^{-1}JM_0 = J^T$.

3. For any $A = MJM^{-1}$: Show that A^T is similar to J^T and so to J and so to A.

<u>Solution</u> (4 points)

1. Suppose that we have one Jordan block J_i. Then

$$\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & \end{pmatrix} \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ & \lambda & 1 & \cdots & 0 \\ & & \lambda & \cdots & 0 \\ & & & \ddots & \\ & & & & \lambda \end{pmatrix} \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & \end{pmatrix} = \begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ 0 & 1 & \lambda & & \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

so J is similar to J^T.

2. Suppose that each J_i satisfies $J_i^T = M_i^{-1} J_i M_i$. Let M_0 be the block-diagonal matrix consisting of the M_i's along the diagonal. Then

$$\begin{aligned} M_0^{-1} J M_0 &= \begin{pmatrix} M_1^{-1} & & & \\ & M_2^{-1} & & \\ & & \ddots & \\ & & & M_n^{-1} \end{pmatrix} \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_n \end{pmatrix} \begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_n \end{pmatrix} \\ &= \begin{pmatrix} M_1^{-1} J_1 M_1 & & & \\ & M_2^{-1} J_2 M_2 & & \\ & & \ddots & \\ & & & M_n^{-1} J_n M_n \end{pmatrix} \\ &= \begin{pmatrix} J_1^T & & & \\ & J_2^T & & \\ & & \ddots & \\ & & & J_n^T \end{pmatrix} = J^T \end{aligned}$$

3. $$A^T = (MJM^{-1})^T = (M^{-1})^T J^T M^T = (M^T)^{-1} J^T (M^T).$$

   So A^T is similar to J^T, which is similar to J, which is similar to A. Thus any matrix is similar to its transpose.

Section 6.6. Problem 20. Why are these statements all true?

(a) If A is similar to B then A^2 is similar to B^2.

(b) A^2 and B^2 can be similar when A and B are not similar.

(c) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ is similar to $\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$.

(d) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ is not similar to $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$.

(e) If we exchange rows 1 and 2 of A, and then exchange columns 1 and 2 the eigenvalues stay the same. In this case $M =$?

Solution (4 points)

(a) If A is similar to B then we can write $A = M^{-1}BM$ for some M. Then $A^2 = M^{-1}B^2M$, so A^2 is similar to B^2.

(b) Let

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $A^2 = B^2$ (so they are obviously similar) but A is not similar to B because nothing but the zero matrix is similar to the zero matrix.

(c)

$$\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(d) These are not similar because the first matrix has a plane of eigenvectors for the eigenvalue 3, while the second only has a line.

(e) In order to exchange two rows of A we multiply on the left by

$$M = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

In order to exchange two columns we multiply on the right by the same M. As $M = M^{-1}$ we see that the new matrix is similar to the old one, so the eigenvalues stay the same.

Section 6.6. Problem 22. If an $n \times n$ matrix A has all eigenvalues $\lambda = 0$ prove that A^n is the zero matrix.

Solution (12 points) Suppose that we have a Jordan block of size i with eigenvalue 0. Then notice that J^2 will have a diagonal of 1's two diagonals above the main diagonal and zeroes elsewhere. J^3 will have a diagonal of 1's three diagonals above the main diagonal, and zeroes elsewhere. Therefore $J^i = 0$, since there is no diagonal i diagonals above the main diagonal. If A has all eigenvalues $\lambda = 0$ then then A is similar to some matrix with Jordan blocks $J_1, \ldots, J_k$ with each J_i of size n_i and $\sum_{i=1}^{k} n_k = n$. Each Jordan block will have eigenvalue 0, so we know that $J_i^{n_i} = 0$, and thus $J_i^n = 0$.

As A^n is similar to a block-diagonal matrix with blocks $J_1^n, J_2^n, \ldots, J_k^n$ and each of these is 0 we know that $A^n = 0$.

Another way to see this is to note that if A has all eigenvalues 0 this means that the characteristic polynomial of A must be x^n, as this is the only polynomial of degree n all of whose roots are 0. Thus $A^n = 0$ by the Cayley-Hamilton theorem.

Section 6.6. Problem 23. For the shifted QR method in the Worked Example 6.6 B, show that A_2 is similar to A_1. No change in eigenvalues, and the A's quickly approach a diagonal matrix.

Solution (12 points) We are asked to show that $A_2 = R_1Q_1 - cs^2I$ is similar to $A_1 = Q_1R_1 - cs^2I$. Note that

$$Q_1A_2Q_1^{-1} = Q_1(R_1Q_1 - cs^2I)Q_1^{-1} = Q_1R_1 - Q_1cs^2IQ_1^{-1} = Q_1R_1 - cs^2I = A_1.$$

Thus A_2 is similar to A_1, and thus their eigenvalues are the same.

Section 6.6. Problem 24. If A is similar to A^{-1}, must all the eigenvalues equal 1 or -1?

Solution (12 points)

No. Consider:

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ is similar to $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1}$.

Section 6.7. Problem 4. Find the eigenvalues and unit eigenvectors of A^TA and AA^T. Keep each $Av = \sigma u$ for the Fibonacci matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Construct the singular value decomposition and verify that A equal s$U\Sigma V^T$.

Solution (4 points)

$$A^TA = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad AA^T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Note that these are the same. (This makes sense, as A is symmetric.) The eigenvalues of this are the roots of $x^2 - 3x + 1$, which are $(3 \pm \sqrt{5})/2$. The unit eigenvectors of this will be

$$\begin{pmatrix} \sqrt{\frac{2}{5-\sqrt{5}}} \\ \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} \\ -\sqrt{\frac{2}{5-\sqrt{5}}} \end{pmatrix}.$$

Then

$$U = \begin{pmatrix} \sqrt{\frac{2}{5-\sqrt{5}}} & -\sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} \\ \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} & \sqrt{\frac{2}{5-\sqrt{5}}} \end{pmatrix} \qquad V = \begin{pmatrix} \sqrt{\frac{2}{5-\sqrt{5}}} & \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} \\ \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} & -\sqrt{\frac{2}{5-\sqrt{5}}} \end{pmatrix}$$

and

$$\Sigma = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \\ & \frac{\sqrt{5}-1}{2} \end{pmatrix}.$$

Section 6.7. Problem 11. Suppose A has orthogonal columns $w_1, \ldots, w_n$ of lengths $\sigma_1, \ldots, \sigma_n$. What are U,Σ and V in the SVD?

Solution (4 points) We will first solve this assuming all of the w_i are nonzero; at the end we will give a modification for the solution in the case that some are 0. As the columns of A are orthogonal we know that A^TA will be a diagonal matrix with diagonal entries $\sigma_1^2, \ldots, \sigma_n^2$. Thus $U = I$ and Σ is the diagonal matrix with entries $\sigma_1, \ldots, \sigma_n$. Then if we define V to be the matrix whose i-th row is the vector w_i/σ_i we will have $A = U\Sigma V^T$, as desired.

Suppose that some of w_i are zero. Take all of the w's that are nonzero and complete them to an orthogonal basis $u_1, \ldots, u_n$ satisfying the conditions that if $w_i \neq 0$ then $u_i = w_i$, and if $w_i = 0$ then $|u_i| = 1$. Then let U, Σ be as above, and V be the matrix whose i-th row is w_i/σ_i if $\sigma_i \neq 0$, and u_i if $\sigma_i = 0$. Then $A = U\Sigma V^T$, as desired.

Section 6.7. Problem 17. The $1, -1$ first difference matrix A has A^TA the second difference matrix. The singular vectors of A are *sine vectors* V and *cosine vectors* u. Then $Av = \sigma u$ is the discrete form of $d/dx(\sin cx) = c(\cos cx)$. This is the best SVD I have seen.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \qquad A^TA = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Then the orthogonal sine matrix is

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} \sin \pi/4 & \sin 2\pi/4 & \sin 3\pi/4 \\ \sin 2\pi/4 & \sin 4\pi/4 & \sin 6\pi/4 \\ \sin 3\pi/4 & \sin 6\pi/4 \sin 9\pi/4 & \end{pmatrix}.$$

(a) Put numbers in V: The unit eigenvectors of A^TA are singular vectors of A. Show that the columns of V have $A^TAv = \lambda v$ with $\lambda = 2 - \sqrt{2}, 2, 2 + \sqrt{2}$.

(b) Multiply AV and verify taht its columns are orthogonal. They are $\sigma_1 u_1$ and $\sigma_2 u_2$ and $\sigma_3 u_3$. The first columns of the cosine matrix U are u_1, u_2, u_3.

(c) Since A is 4×3 weneed a fourth orthogonal vector u_4. It comes from the nullspace of A^T. What is u_4?

Solution (12 points)

(a) We are asked to show that the columns of V are eigenvectors of A^TA. The characteristic polynomial of A^TA is $x^3 - 6x^2 + 10x - 4$, which can be factored as $(x-2)(x^2 - 4x + 2)$. By the quadratic formua the roots of this are exactly the eigenvalues specified.

Note that

$$V = \begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix}.$$

Then note that the three vectors $\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$ are scalar multiples of the columns of V, and it is easy to check that they are indeed eigenvectors of A^TA with the right eigenvalues.

(b)

$$AV = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2}-1 & -\sqrt{2} & -\sqrt{2}-1 \\ 1-\sqrt{2} & -\sqrt{2} & 1+\sqrt{2} \\ -1 & \sqrt{2} & -1 \end{pmatrix}.$$

It is easy to check that these columns are orthogonal.

(c) Note that $A^T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$. The nullspace of this is generated by $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

Section 8.5. Problem 4. The first three *Legendre polynomials* are $1, x, x^2 - 1/3$. Choose c so that the fourth polynomial $x^3 - cx$ is orthogonal to the first three. All integrals go from -1 to 1.

Solution (4 points) We compute

$$\int_{-1}^{1} x^3 - cx\,dx = 0 \qquad \int_{-1}^{1} (x^3 - cx)x\,dx = \frac{2}{5} - \frac{2}{3}c \qquad \int_{-1}^{1} (x^3 - cx)(x^2 - \frac{1}{3})\,dx = 0.$$

Thus in order for $x^3 - cx$ to be orthogonal to the other three we need $c = 3/5$.

Section 8.5. Problem 5. For the square wave $f(x)$ in Example 3 show that

$$\int_{0}^{2\pi} f(x)\cos x\,dx = 0 \qquad \int_{0}^{2\pi} f(x)\sin x\,dx = 4 \qquad \int_{0}^{2\pi} f(x)\sin 2x\,dx = 0.$$

Which three Fourier coefficients come from those integrals?

Solution (4 points) By definition, coefficients that come from these are a_1, b_1, b_2, respectively. We compute

$$\begin{aligned} \int_{0}^{2\pi} f(x)\cos x\,dx &= \int_{0}^{\pi} \cos x\,dx - \int_{\pi}^{2\pi} \cos x\,dx = 0 \\ \int_{0}^{2\pi} f(x)\sin x\,dx &= \int_{0}^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx = 4 \\ \int_{0}^{2\pi} f(x)\sin 2x\,dx &= \int_{0}^{\pi} \sin 2x\,dx - \int_{\pi}^{2\pi} \sin 2x\,dx = 0. \end{aligned}$$

Section 8.5. Problem 12. The functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$ are a basis for a Hilberts space. Write the derivatives of those first five functions as combinations of the same five functions. What is the 5×5 "differentiation matrix" for those functions?

Solution (12 points)

We know that $1' = 0$, and that

$$(\cos x)' = -\sin x \qquad (\sin x)' = \cos x \qquad (\cos 2x)' = -2\sin 2x \qquad (\sin 2x)' = 2\cos 2x.$$

Thus the "differentiation matrix" is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

Section 8.5. Problem 13. Find the Fourier coefficients a_k and b_k of the square pulse $F(x)$ centered at $x = 0$: $f(x) = 1/h$ for $|x| \le h/2$ and $F(x) = 0$ for $h/2 < |x| \le \pi$. As $h \to 0$, this $F(x)$ approaches a delta function. Find the limits of a_k and b_k.

Solution (12 points) We compute

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,dx = \frac{1}{h\pi}\int_{-h/2}^{h/2} 1\,dx = \frac{1}{\pi}. \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos kx\,dx = \frac{1}{\pi h}\int_{-h/2}^{h/2} \cos kx\,dx = \frac{1}{\pi hk}\sin kx\Big|_{-h/2}^{h/2} = \frac{2}{\pi hk}\sin\frac{kh}{2}. \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin kx\,dx = \frac{1}{\pi h}\int_{-h/2}^{h/2} \sin kx\,dx = \frac{1}{\pi k}\cos kx\Big|_{-h/2}^{h/2} = 0. \end{aligned}$$

Thus as $h \to 0$ we see that $a_0 \to 1/\pi$ and $b_k \to 0$. We also compute

$$\lim_{h\to 0} a_k = \lim_{h\to 0} \frac{1}{\pi}\frac{2}{hk}\sin\frac{hk}{2} = \frac{1}{\pi}\lim_{x\to 0}\frac{\sin x}{x} = \frac{1}{\pi}$$

where we set $x = hk/2$.