18.06 Quiz 3 Solutions May 8, 2010 Professor Strang

Your PRINTED name is: ____________________ **1.**

Your recitation number is ____________________ **2.**

3.

1. **(40 points)** Suppose u is a unit vector in R^n, so $u^Tu = 1$. This problem is about the n by n symmetric matrix $H = I - 2u\,u^T$.

   (a) Show directly that $H^2 = I$. Since $H = H^T$, we now know that H is not only

   symmetric but also ____________________.

   Solution Explicitly, we find $H^2 = (I - 2uu^T)^2 = I^2 - 4uu^T + 4uuu^Tuu^T$ (2 points): since $u^Tu = 1$, $H^2 = I$ (3 points). Since $H = H^T$, we also have $H^TH = 1$, implying that H is an orthogonal (or unitary) matrix.

   (b) One eigenvector of H is u itself. Find the corresponding eigenvalue.

   Solution Since $Hu = (I - 2uu^T)u = u - 2uu^Tu = u - 2u = -u$, $\lambda = -1$.

   (c) If v is any vector perpendicular to u, show that v is an eigenvector of H and **find the eigenvalue.** With all these eigenvectors v, that eigenvalue must be repeated how many times? Is H **diagonalizable**? Why or why not?

   Solution For any vector v orthogonal to u (i.e. $u^Tv = 0$), we have $Hv = (I-2uu^T)v = v - 2uu^Tv = v$, so the associated λ is 1. The orthogonal complement to the space spanned by u has dimension $n-1$, so there is a basis of $(n-1)$ orthonormal eigenvectors with this eigenvalue. Adding in the eigenvector u, we find that H is diagonalizable.

   (d) Find the diagonal entries H_{11} and H_{ii} in terms of $u_1, \ldots, u_n$. Add up $H_{11} + \ldots + H_{nn}$ and separately add up the eigenvalues of H.

   Solution Since ith diagonal entry of uu^T is u_i^2, the i diagonal entry of H is $H_{ii} = 1-2u_i^2$ (3 points). Summing these together gives $\sum_{i=1}^n H_{ii} = n - 2\sum_{i=1}^n u_i^2 = n - 2$ (3 points). Adding up the eigenvalues of H also gives $\sum \lambda_i = (1) - 1 + (n-1)(1) = n - 2$ (4 points).

2. **(30 points)** Suppose A is a positive definite symmetric n by n matrix.

(a) How do you know that A^{-1} is also positive definite? (We know A^{-1} is symmetric. I just had an e-mail from the International Monetary Fund with this question.)

Solution Since a matrix is positive-definite if and only if all its eigenvalues are positive (5 points), and since the eigenvalues of A^{-1} are simply the inverses of the eigenvalues of A, A^{-1} is also positive definite (the inverse of a positive number is positive) (5 points).

(b) Suppose Q is any **orthogonal** n by n matrix. How do you know that $Q\,A\,Q^T = Q\,A\,Q^{-1}$ is positive definite? Write down which test you are using.

Solution Using the energy text ($x^T Ax > 0$ for nonzero x), we find that $x^T QAQ^T x = (Q^T x)^T A(Q^T x) > 0$ for all nonzero x as well (since Q is invertible). Using the positive eigenvalue test, since A is similar to QAQ^{-1} and similar matrices have the same eigenvalues, QAQ^{-1} also has all positive eigenvalues. (5 points for test, 5 points for application)

(c) Show that the block matrix

$$B = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$$

is positive **semidefinite**. How do you know B is not positive definite?

Solution First, since B is singular, it cannot be positive definite (it has eigenvalues of 0). However, the pivots of B are the pivots of A in the first n rows followed by 0s in the remaining rows, so by the pivot test, B is still semi-definite. Similarly, the first n upper-left determinants of B are the same as those of A, while the remaining ones are 0s, giving another proof. Finally, given a nonzero vector

$$u = \begin{bmatrix} x \\ y \end{bmatrix}$$

where x and y are vectors in $\mathbf{R}^n$, one has $u^T Bu = (x+y)^T A(x+y)$ which is nonnegative (and zero when $x + y = 0$).

3. **(30 points)** This question is about the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}$$

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(a) Find its eigenvalues and eigenvectors.

Write the vector $u(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ as a combination of those eigenvectors.

Solution Since $\det(A - \lambda I) = \lambda^2 + 4$, the eigenvalues are $2i, -2i$ (4 points). Two associated eigenvectors are $[1 \quad -2i]^T, [1 \quad 2i]^T$, though there are many other choices (4 points). $u(0)$ is just the sum of these two vectors (2 points).

(b) Solve the equation $\frac{du}{dt} = Au$ starting with the same vector $u(0)$ at time $t = 0$.

In other words: the solution $u(t)$ is what combination of the eigenvectors of A?

Solution One simply adds in factors of $e^{\lambda t}$ to each term, giving

$$u(t) = e^{2it} \begin{bmatrix} 1 \\ -2i \end{bmatrix} + e^{-2it} \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

(c) Find the 3 matrices in the Singular Value Decomposition $A = U \Sigma V^T$ in two steps.

–First, compute V and Σ using the matrix $A^T A$.

–Second, find the (orthonormal) columns of U.

Solution Note that $A^T A = V\Sigma^T U^T U \Sigma V^T = V \Sigma^2 V^T$, so the diagonal entries of Σ are simply the positive roots of the eigenvalues of

$$A^T A = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $\sigma_1 = 4, \sigma_2 = 1$. Since $A^T A$ is already diagonal, V is the identity matrix. The columns of U should satisfy $Au_1 = \sigma_1 v_1, Au_2 = \sigma_2 v_2$: by inspection, one obtains

$$u_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$