18.06 Quiz 1 March 1, 2010 Professor Strang

Your PRINTED name is: ____________________ **1.**

Your recitation number or instructor is ____________________ **2.**

3.

4.

1. Forward elimination changes $A\mathbf{x} = \mathbf{b}$ to a row reduced $R\mathbf{x} = \mathbf{d}$: the complete solution is

$$\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \mathbf{c_1} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \mathbf{c_2} \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

(a) **(14 points)** What is the 3 by 3 reduced row echelon matrix R and what is $\mathbf{d}$?

Solution: First, since R is in reduced row echelon form, we must have

$$\boxed{\mathbf{d} = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix}^T}$$

The other two vectors provide special solutions for R, showing that R has rank 1: again, since it is in reduced row echelon form, the bottom two rows must be all 0, and the top row is $\begin{bmatrix} 1 & -2 & -5 \end{bmatrix}^T$, i.e. $\boxed{R = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$.

(b) **(10 points)** If the process of elimination subtracted 3 times row 1 from row 2 and then 5 times row 1 from row 3, what matrix connects R and $\mathbf{d}$ to the original A and $\mathbf{b}$? Use this matrix to find A and $\mathbf{b}$.

Solution: The matrix connecting R and $\mathbf{d}$ to the original A and $\mathbf{b}$ is

$$E = E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}}$$

That is, $R = EA$ and $E\mathbf{b} = \mathbf{d}$. Thus, $A = E^{-1}R$ and $\mathbf{b} = E^{-1}\mathbf{d}$, giving

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -2 & -5 \\ 3 & -6 & -15 \\ 5 & -10 & -25 \end{bmatrix}}$$

$$\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 12 \\ 20 \end{bmatrix}}$$

2. Suppose A is the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix}.$$

(a) **(16 points)** Find all special solutions to $Ax = 0$ and describe in words the whole nullspace of A.

Solution: First, by row reduction

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the special solutions are

$$\boxed{s_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} 0 \\ -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}}$$

Thus, $N(A)$ is a plane in $\mathbb{R}^4$ given by all linear combinations of the special solutions.

(b) **(10 points)** Describe the column space of this particular matrix A. "All combinations of the four columns" is not a sufficient answer.

Solution: $C(A)$ is a plane in $\mathbb{R}^3$ given by all combinations of the pivot columns, namely

$$\boxed{c_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix}}$$

(c) **(10 points)** What is the reduced row echelon form $R^* = \text{rref}(B)$ when B is the 6 by 8 block matrix

$$B = \begin{bmatrix} A & A \\ A & A \end{bmatrix} \text{ using the same } A?$$

Solution: Note that B immediately reduces to

$$B = \begin{bmatrix} A & A \\ 0 & 0 \end{bmatrix}$$

We reduced A above: the row reduced echelon form of of B is thus

$$\boxed{B = \begin{bmatrix} rref(A) & rref(A) \\ 0 & 0 \end{bmatrix}, rref(A) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

3. **(16 points) Circle the words** that correctly complete the following sentence:

(a) Suppose a 3 by 5 matrix A has rank $r = 3$. Then the equation $Ax = b$

(always / sometimes but not always)

has (a unique solution / many solutions / no solution).

Solution: the equation $Ax = b$ $\boxed{\text{always}}$ has $\boxed{\text{many solutions}}$.

(b) What is the column space of A? Describe the nullspace of A.

Solution: $\boxed{\text{The column space is a 3-dimensional space inside a 3-dimensional space}}$, i.e. it contains all the vectors, and $\boxed{\text{the nullspace has dimension } 5 - 3 = 2 > 0 \text{ inside } \mathbb{R}^5.}$

4. Suppose that A is the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 4 \end{bmatrix}.$$

(a) **(10 points)** Explain in words how knowing all solutions to $A\mathbf{x} = \mathbf{b}$ decides if a given vector $\mathbf{b}$ is in the column space of A.

Solution: The column space of A contains all linear combinations of the columns of A, which are precisely vectors of the form $A\mathbf{x}$ for an arbitrary vector $\mathbf{x}$. Thus, $\boxed{A\mathbf{x} = \mathbf{b} \text{ has a solution if and only if } \mathbf{b} \text{ is in the column space of } A}$.

(b) **(14 points)** Is the vector $\mathbf{b} = \begin{bmatrix} 8 \\ 28 \\ 14 \end{bmatrix}$ in the column space of A?

Solution: $\boxed{\text{Yes}}$. Reducing the matrix combining A and $\mathbf{b}$ gives

$$\left[\begin{array}{cc|c} 2 & 1 & 8 \\ 6 & 5 & 28 \\ 2 & 4 & 14 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 8 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 8 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$, and $\mathbf{b}$ is in the column space of A.