18.06 Final Exam May 18, 2010 Professor Strang

1. **(12 points)** This question is about the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{bmatrix}.$$

(a) Find a lower triangular L and an upper triangular U so that $A = LU$.

(b) Find the reduced row echelon form $R = rref(A)$. How many independent columns in A?

(c) Find a basis for the nullspace of A.

(d) If the vector b is the sum of the four columns of A, write down the complete solution to $Ax = b$.

2. **(11 points)** This problem finds the curve $y = C + D\,2^t$ which gives the best least squares fit to the points $(t, y) = (0, 6), (1, 4), (2, 0)$.

(a) Write down the 3 equations that would be satisfied *if* the curve went through all 3 points.

(b) Find the coefficients C and D of the best curve $y = C + D2^t$.

(c) What values should y have at times $t = 0, 1, 2$ so that the best curve is $y = 0$?

3. **(11 points)** Suppose $Av_i = b_i$ for the vectors $v_1, \ldots, v_n$ and $b_1, \ldots, b_n$ in R^n. Put the v's into the columns of V and put the b's into the columns of B.

(a) Write those equations $Av_i = b_i$ in matrix form. *What condition on which vectors* allows A to be determined uniquely? Assuming this condition, *find A from V and B.*

(b) Describe the column space of that matrix A in terms of the given vectors.

(c) What additional condition on which vectors makes A an *invertible* matrix? Assuming this, find A^{-1} from V and B.

4. **(11 points)**

(a) Suppose x_k is the fraction of MIT students who prefer calculus to linear algebra at year k. The remaining fraction $y_k = 1 - x_k$ prefers linear algebra.

At year $k+1$, 1/5 of those who prefer calculus change their mind (possibly after taking 18.03). Also at year $k+1$, 1/10 of those who prefer linear algebra change their mind (possibly because of this exam).

Create the matrix A to give $\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = A \begin{bmatrix} x_k \\ y_k \end{bmatrix}$ and find the limit of $A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as $k \to \infty$.

(b) Solve these differential equations, starting from $x(0) = 1, \quad y(0) = 0$:

$$\frac{dx}{dt} = 3x - 4y \quad \frac{dy}{dt} = 2x - 3y\,.$$

(c) For what initial conditions $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$ does the solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ to this differential equation lie on a single straight line in R^2 for all t?

5. **(11 points)**

(a) Consider a $120°$ rotation around the axis $x = y = z$. Show that the vector $i = (1, 0, 0)$ is rotated to the vector $j = (0, 1, 0)$. (Similarly j is rotated to $k = (0, 0, 1)$ and k is rotated to i.) How is $j - i$ related to the vector $(1, 1, 1)$ along the axis?

(b) Find the matrix A that produces this rotation (so Av is the rotation of v). Explain why $A^3 = I$. What are the eigenvalues of A?

(c) If a 3 by 3 matrix P projects every vector onto the plane $x+2y+z = 0$, find three eigenvalues and three independent eigenvectors of P. No need to compute P.

6. **(11 points)** This problem is about the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}.$$

(a) Find the eigenvalues of $A^T A$ and also of AA^T. For both matrices find a complete set of orthonormal eigenvectors.

(b) If you apply the Gram-Schmidt process (orthonormalization) to the columns of this matrix A, what is the resulting output?

(c) If A is *any* m by n matrix with $m > n$, tell me why AA^T cannot be positive definite. Is A^TA always positive definite? (If not, what is the test on A?)

7. **(11 points)** This problem is to find the determinants of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} x & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

(a) Find $\det A$ and give a reason.

(b) Find the cofactor C_{11} and then find $\det B$. This is the volume of what region in R^4?

(c) Find $\det C$ for any value of x. You could use linearity in row 1.

8. **(11 points)**

(a) When A is similar to $B = M^{-1}AM$, prove this statement:

If $A^k \to 0$ when $k \to \infty$, then also $B^k \to 0$.

(b) Suppose S is a fixed invertible 3 by 3 matrix.

This question is about all the matrices A that are diagonalized by S, so that $S^{-1}AS$ is diagonal. Show that these matrices A form a subspace of 3 by 3 matrix space. (Test the requirements for a subspace.)

(c) Give a basis for the space of 3 by 3 *diagonal matrices.* Find a basis for the space in part (b) — all the matrices A that are diagonalized by S.

9. **(11 points)** This square network has 4 nodes and 6 edges. On each edge, the direction of positive current $w_i > 0$ is from lower node number to higher node number. The voltages at the nodes are $(v_1, v_2, v_3, v_4.)$

(a) Write down the incidence matrix A for this network (so that Av gives the 6 voltage differences like $v_2 - v_1$ across the 6 edges). What is the rank of A? What is the dimension of the nullspace of A^T?

(b) Compute the matrix $A^T A$. What is its rank? What is its nullspace?

(c) Suppose $v_1 = 1$ and $v_4 = 0$. If each edge contains a unit resistor, the currents $(w_1, w_2, w_3, w_4, w_5, w_6)$ on the 6 edges will be $w = -Av$ by Ohm's Law. Then Kirchhoff's Current Law (flow in = flow out at every node) gives $A^T w = 0$ which means $\boldsymbol{A^T A v = 0}$. Solve $A^T Av = 0$ for the unknown voltages v_2 and v_3. Find all 6 currents w_1 to w_6. How much current enters node 4?