18.06 Quiz 3 Solution

Hold on Friday, 1 May 2009 at 11am in Walker Gym.

Total: 65 points.

Problem 1:

For each part, give **as much information as possible** about the **eigenvalues** of the matrix A described in that part. (Each part describes a *different* matrix A. A may be complex.)

(a) The recurrence $\mathbf{u}_{k+1} = A\mathbf{u}_k$ has a solution where $\|\mathbf{u}_k\| \to 0$ as $k \to \infty$ for one initial vector $\mathbf{u}_0$, but also has a solution with $\|\mathbf{u}_k\| \to \infty$ as $k \to \infty$ for a *different* choice of the initial vector $\mathbf{u}_0$.

(b) The equation $(A^2 - 4I)\mathbf{x} = \mathbf{b}$ has no solution for some right-hand side $\mathbf{b}$.

(c) $A = e^{B^{\mathrm{T}}B}$ for some real matrix B with full column rank.

(d) $A = B^{\mathrm{T}}B$ for a 4×3 real matrix B, and the matrix BB^{T} has eigenvalues $\lambda = 3, 2, 1, 0$. (Hint: think about the SVD of B.)

Solution (20 points = 5+5+5+5)

(a) (There was a bug in this problem: in the first condition, we should have required the initial vector $\mathbf{u}_0$ to be nonzero.) The first condition implies that A has an eigenvalue with absolute value $|\lambda| < 1$. The second condition implies that either A has an eigenvalue with absolute value $|\lambda| > 1$, or A is defective for 2 eigenvalues λ with $|\lambda| = 1$.

(b) The condition says that $A^2 - 4I$ is singular. But we know that, if $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A, then the eigenvalues of $A^2 - 4I$ are $\lambda_1^2 - 4, \ldots, \lambda_n^2 - 4$. The condition $A^2 - 4I$ being singular says that one of $\lambda_i^2 - 4$ is zero, and hence $\lambda_i = 2$ or -2. That is to say A has an eigenvalue 2 or -2.

(c) Since B has full column rank, the eigenvalues of $B^{\mathrm{T}}B$ are postive real numbers λ_i. Hence, we know $A = e^{B^{\mathrm{T}}B}$ has eigenvalues e^{λ_i}; they are real numbers bigger than 1.

(d) Since BB^{T} and $B^{\mathrm{T}}B$ have the same set of *nonzero* eigenvalues. So $B^{\mathrm{T}}B$ must have eigenvalues $3, 2, 1$. Moreover, since B is a 4×3 matrix, $B^{\mathrm{T}}B$ is a 3×3 matrix. Hence, $3, 2, 1$ are exactly all the eigenvalues.

Problem 2: You are given the matrix

$$A = \begin{pmatrix} 0.5 & 0.2 & 0.2 \\ 0.1 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}.$$

(i) What are the eigenvalues of A? [*Hint:* Very little calculation required! You should be able to see two eigenvalues by inspection of the form of A, and the third by an easy calculation. You *shouldn't* need to compute $\det(A - \lambda I)$ unless you really want to do it the hard way.]

(ii) The vector $\mathbf{u}(t)$ solves the system

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}$$

for some initial condition $\mathbf{u}(0)$. If you are told that $\mathbf{u}(t)$ approaches some constant vector as $t \to \infty$, give as much true information as possible regarding the initial condition $\mathbf{u}(0)$.

[*Note:* be sure you understand that this is *not the same thing* as solving the recurrence $\mathbf{u}_{k+1} = A\mathbf{u}_k$! Imagine how you would find $\mathbf{u}(t)$ if you knew what $\mathbf{u}(0)$ was.]

Solution (20 points = 10+10)

(i) First, the last two columns of A are the same. Hence A is singular and it must have an eigenvalue $\lambda_1 = 0$. Also, we observe that A is a Markov matrix. This means that $\lambda_2 = 1$ is an eigenvalue of A. Finally, we know the trace of A is the sum of its three eigenvalues. So, $\mathrm{Tr}(A) = 0.5 + 0.5 + 0.3 = 1.3$ and the last eigenvalue is $\lambda_3 = 1.3 - 1 - 0 = 0.3$.

(ii) We can write $\mathbf{u}(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ using three eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, which correspond to $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 0.3$, respectively. We know that this system has solution $\mathbf{u}(t) = c_1\mathbf{v}_1 + c_2 e^t \mathbf{v}_2 + c_3 e^{0.3t}\mathbf{v}_3$. So, if either one of c_2 and c_3 is nonzero, the system would blow up as $t \to \infty$. Therefore, the only possibility for $\mathbf{u}(t)$ to approaches some constant is to have $c_2 = c_3 = 0$, that is to say that $\mathbf{u}(0)$ is a multiple of the eigenvector $\mathbf{v}_1 = (0, -1, 1)^{\mathrm{T}}$. In this case, $\mathbf{u}(t) = \mathbf{u}(0) = c_1\mathbf{v}_1$ is a constant.

Problem 3:

The 3×3 matrix A has three independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ with corresponding eigenvalues λ_1, λ_2, and λ_3 (that is, $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for $i = 1, 2, 3$).

If

$$\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

for some coefficients c_1, c_2, and c_3, then write (in terms of λ_i, c_i, and $\mathbf{v}_i$) a formula for the solution $\mathbf{x}$ of

$$A^2\mathbf{x} + 2A\mathbf{x} - 3I\mathbf{x} = \mathbf{b}$$

(you can assume that a solution exists for any $\mathbf{b}$).

Solution (10 points)

Using the eigenvalues $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, we have

$$\begin{aligned}\mathbf{x} &= (A^2 + 2A - 3I)^{-1}\mathbf{b} \\ &= (A^2 + 2A - 3I)^{-1}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) \\ &= \frac{c_1}{\lambda_1^2 + 2\lambda_1 - 3}\mathbf{v}_1 + \frac{c_2}{\lambda_2^2 + 2\lambda_2 - 3}\mathbf{v}_2 + \frac{c_3}{\lambda_3^2 + 2\lambda_3 - 3}\mathbf{v}_3.\end{aligned}$$

Problem 4: A is a 3×3 real-symmetric matrix. Two of its eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$ with eigenvectors $\mathbf{v}_1 = (1,1,1)^{\mathrm{T}}$ and $\mathbf{v}_2 = (1,-1,0)^{\mathrm{T}}$, respectively. The third eigenvalue is $\lambda_3 = 0$.

(I) Give an eigenvector $\mathbf{v}_3$ for the eigenvalue λ_3. (*Hint:* what must be true of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$?)

(II) Using your result from (I), write the matrix e^A as the product of three matrices, and explicitly give the three matrices. (You need not work out the arithmetic, but your answer should contain no matrix inverses or matrix exponentials. *If you find yourself doing a lot of arithmetic, you are forgetting a useful property of this matrix!*)

Solution (15 points = 7+8)

(I) For a real-symmetric matrix, its eigenvectors are orthogonal to each other. So, by inspection, in order for $\mathbf{v}_3$ to be perpendicular to $\mathbf{v}_2$, we need its first two components same. Hence, we should take $\mathbf{v}_3$ to be $(1,1,-2)^{\mathrm{T}}$. To easy the second part, we can normalize the eigenvectors

$$\begin{aligned}
\mathbf{q}_1 &= \mathbf{v}_1/\|\mathbf{v}_1\| = (1,1,1)^{\mathrm{T}}/\sqrt{3}, \\
\mathbf{q}_2 &= \mathbf{v}_2/\|\mathbf{v}_2\| = (1,-1,0)^{\mathrm{T}}/\sqrt{2}, \\
\mathbf{q}_3 &= \mathbf{v}_3/\|\mathbf{v}_3\| = (1,1,-2)^{\mathrm{T}}/\sqrt{6}.
\end{aligned}$$

Alternatively, we can use Gram-Schmidt to find (a multiple of) $\mathbf{v}_3$ as follows. We start with $\mathbf{v} = (1,0,0)$,

$$\mathbf{v}_3 = \mathbf{v} - (\mathbf{v}\cdot\mathbf{q}_1)\mathbf{q}_1 - (\mathbf{v}\cdot\mathbf{q}_2)\mathbf{q}_2 = (1,0,0)^{\mathrm{T}} - \frac{1}{3}(1,1,1)^{\mathrm{T}} - \frac{1}{2}(1,-1,0)^{\mathrm{T}} = (\frac{1}{6},\frac{1}{6},-\frac{1}{3})^{\mathrm{T}}.$$

(II) We can write

$$\begin{aligned}
A &= S\Lambda S^{-1} = Q\Lambda Q^{\mathrm{T}} \\
&= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix}.
\end{aligned}$$

Hence,

$$\begin{aligned}
e^A &= Qe^{\Lambda}Q^{\mathrm{T}} \\
&= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} e & 0 & 0 \\ 0 & 1/e & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix}.
\end{aligned}$$