18.06 Quiz 2 Solution

Hold on Wednesday, 1 April 2009 at 11am in Walker Gym.
Total: 70 points.

Problem 1:

(a) If P is the projection matrix onto the *null* space of A, then $P\mathbf{y} - \mathbf{y}$, for any $\mathbf{y}$, is in the ____________________ space of A.

(b) If $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$, then the closest vector to $\mathbf{b}$ in $N(A^{\mathrm{T}})$ is ____________________ (best answer).

(c) If the *rows* of A (an $m \times n$ matrix) are independent, then the dimension of $N(A^{\mathrm{T}}A)$ is __________.

(d) If a matrix U has orthonormal *rows*, then $I =$ __________ and the projection matrix onto the *row* space of U is __________. (Your answers should be the simplest expressions involving U and U^{T} only.)

Solution (20 points = 5+5+5+5)

Answers: (a) row; (b) 0; (c) $n - m$; (d) UU^{T}, $U^{\mathrm{T}}U$.

(a) Since $P\mathbf{y}$ is the projection to the nullspace of A, $P\mathbf{y} - \mathbf{y}$ is orthogonal to the null space; it then must lie in the row space of A.

(b) Since $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$, $\mathbf{b}$ is in the column space $C(A)$ of A. We know that the left nullspace $N(A^{\mathrm{T}})$ is orthogonal to the column space. So the closest vector to $\mathbf{b}$ is 0.

(c) We derived in Problem 7 of Pset 4 that the nullspace of $A^{\mathrm{T}}A$ is the same as the nullspace of A; the latter has dimension $n - m$ because the matrix A is of full row rank m.

Alternatively, we also derived the following in lecture, and it is in the text, and on the practice-exam handout: the ranks of A and $A^{\mathrm{T}}A$ are the same, so both equal to m. Since A has full row rank and $A^{\mathrm{T}}A$ has n columns, $N(A^{\mathrm{T}}A)$ has dimension $n - m$.

(d) Note that $U^{\mathrm{T}} = Q$, a matrix with orthonormal columns. We saw in class that $I = Q^{\mathrm{T}}Q = UU^{\mathrm{T}}$, and the projection matrix onto $C(Q) = C(U^{\mathrm{T}})$ is $QQ^{\mathrm{T}} = U^{\mathrm{T}}U$.

Problem 2: The matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & -7 \\ 2 & 4 & 1 & -5 \\ 1 & 2 & 2 & -16 \end{pmatrix}$$

is converted to row-reduced echelon form by the usual row-elimination steps, resulting in the matrix:

$$R = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(♣) The *minimum* number of columns of A that form a *dependent* set of vectors is __________. The *maximum* number of columns of A that forms an *independent* set of vectors is __________.

(♢) Give an *orthonormal* basis for the *row* space of A. (Careful: be sure you start with a basis for the row space, not containing any dependent vectors.) Your answer may contain square roots left as $\sqrt{\textit{some number}}$.

(♠) Given the vector $\mathbf{b} = \begin{pmatrix} 2 & 5 & -9 & 3 \end{pmatrix}^{\mathrm{T}}$, compute the *closest* vector $\mathbf{p}$ to $\mathbf{b}$ in the *row space* $C(A^{\mathrm{T}})$? (Hint: less calculation is needed if you use your answer from ♢.)

(♡) In terms of your answer $\mathbf{p}$ to ♠ above, what is the closest vector to $\mathbf{b}$ in the *nullspace* $N(A)$? (No calculation required, and you need not have solved ♠: you can leave your answer in terms of $\mathbf{p}$ and $\mathbf{b}$.)

Solution (30 points = 6+10+10+4)

Answers: (♣) 2, 2; (♢) see below; (♠) $\mathbf{p} = (2 \;\; 4 \;\; 0 \;\; 4)^{\mathrm{T}}$; (♡) $\mathbf{b} - \mathbf{p}$.

(♣) The key point of the problem is that the dependency of columns in R and A is the same. By inspection of R (or A), the first two columns are dependent, so that is the smallest dependent set. R has two pivots, so A is rank 2 and the column space is 2-dimensional, so 2 is the maximum number of independent columns. Equivalently, the maximum number of independent columns is the number of columns in any basis for $C(A)$, such as the 2 pivot columns.

(♢) Note that the (row-reduced) echelon form R has the same row space as A. We may therefore start Gram-Schmidt on the pivot rows of R, which form a basis for the row space of R and A.

$$\mathbf{q}_1 = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} = (\frac{1}{3} \;\; \frac{2}{3} \;\; 0 \;\; \frac{2}{3})^{\mathrm{T}}$$

$$\mathbf{q}_2 = \frac{\mathbf{a}_2 - \mathbf{q}_1^{\mathrm{T}}\mathbf{a}_2\mathbf{q}_1}{\|\mathbf{a}_2 - \mathbf{q}_1^{\mathrm{T}}\mathbf{a}_2\mathbf{q}_1\|} = \frac{(0 \;\; 0 \;\; 1 \;\; -9)^{\mathrm{T}} + 6 \cdot (\frac{1}{3} \;\; \frac{2}{3} \;\; 0 \;\; \frac{2}{3})^{\mathrm{T}}}{\|(0 \;\; 0 \;\; 1 \;\; -9)^{\mathrm{T}} + 6 \cdot (\frac{1}{3} \;\; \frac{2}{3} \;\; 0 \;\; \frac{2}{3})^{\mathrm{T}}\|}$$

$$= \frac{(2 \;\; 4 \;\; 1 \;\; -5)^{\mathrm{T}}}{\|(2 \;\; 4 \;\; 1 \;\; -5)^{\mathrm{T}}\|} = (2 \;\; 4 \;\; 1 \;\; -5)^{\mathrm{T}} / \sqrt{46}.$$

REMARK: One can also obtain an orthonormal basis by starting with 2 rows of A since in this case any 2 rows are independent and form a basis. But the pivot rows of R are a nicer basis (more zeros), and the calculations are therefore much simpler.

(♠) The closest vector $\mathbf{p}$ should be given by the projection to the row space. That is

$$\mathbf{p} = \mathbf{p}^{\mathrm{T}}\mathbf{q}_1\mathbf{q}_1 + \mathbf{p}^{\mathrm{T}}\mathbf{q}_2\mathbf{q}_2 = 6 \cdot (\frac{1}{3} \;\; \frac{2}{3} \;\; 0 \;\; \frac{2}{3})^{\mathrm{T}} + 0 = (2 \;\; 4 \;\; 0 \; 4)^{\mathrm{T}}.$$

(♡) The closest vector $\mathbf{p}$ of $\mathbf{b}$ in the row space is exactly the projection in the row space. But the row space and the nullspace are orthogonal to each other. Then, $\mathbf{b} - \mathbf{p}$ is exactly the orthogonal projection in the nullspace $N(A)$; it is the closest vector to $\mathbf{b}$ in the nullspace.

Problem 3: You are told that the least-square linear fit to three points $(0, b_1)$, $(1, b_2)$, and $(2, b_3)$ is $C + Dt$ for $C = 1$ and $D = -2$. That is, the fit is $1 - 2t$.

In this question, you will work backwards from this fit to reason about the unknown values $\mathbf{b} = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}^T$ at the coordinates $t = 0, 1, 2$.

(i) Write down the explicit equations that $\mathbf{b}$ must satisfy for $1 - 2t$ to be the least-square linear fit. (The points do *not* have to fall exactly on the line.)

(ii) If all the points fall *exactly* on the line $1 - 2t$, then $\mathbf{b} =$ __________. Check that this satisfies your equations in (i).

(iii) More generally, if all the points fall exactly on *any* straight line, then $\mathbf{b}$ is in the ______________ space of what matrix? (Write down the matrix.)

Solution (20 points = 10+5+5)

Answers: (i) see below; (ii) $\mathbf{b} = (1 \ \ -1 \ \ -3)$; (iii) see below.

(i) The system that we would solve if the line passed exactly through all of the points is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

However, since the line may not pass through all the points this system may have no solution, and instead we find the least-square solution by solving the normal equations:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

That is

$$\begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} b_1 + b_2 + b_3 \\ b_2 + 2b_3 \end{pmatrix}$$

Since the least-square fit is $1-2t$, the above linear system has solution $(1 \ \ -2)^{\mathrm{T}}$. Hence, b_1, b_2, b_3 should satisfy

$$\begin{aligned} b_1 + b_2 + b_3 &= -3 \\ b_2 + 2b_3 &= -7 \end{aligned}$$

(ii) If all the points fall exactly on the line $1 - 2t$,

$$b_1 = 1 - 2 \cdot 0 = 1, \quad b_2 = 1 - 2 \cdot 1 = -1, \quad b_3 = 1 - 2 \cdot 2 = -3.$$

We plug the solution back in the relations above and check.

$$1 - 1 - 3 = -3, \quad -1 + 2 \times (-3) = -7.$$

(iii) If all points fall exactly on a straight line, the following system would have a solution.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

In other words, $\mathbf{b}$ lies in the column space of the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}.$$