18.06 Quiz 1 Solution

Hold on Monday, 2 March 2009 at 11am in Walker Gym.

Total: 60 points.

Problem 1: Your classmate, Nyarlathotep, performed the usual elimination steps to convert A to echelon form U, obtaining:

$$U = \begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find a set of vectors spanning the nullspace $N(A)$.

(b) If $U\vec{y} = \begin{pmatrix} 9 \\ -12 \\ 0 \end{pmatrix}$, find the complete solution $\vec{y}$ (i.e. describe all possible solutions $\vec{y}$).

(c) Nyarla gave you a matrix

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}$$

and told you that $A = LU$. Describe the complete sequence of elimination steps that Nyarla performed, assuming that she did elimination in the usual way starting with the first column and eliminating downwards. That is, Nyarla first subtracted ______ times the first row from the second row, then subtracted ______ times the first row from the third row, then subtracted ____________________________________. (Be careful about signs: *adding* a multiple of a row is the same as subtracting a *negative* multiple of that row.)

(d) If $A\vec{x} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$, then $U\vec{x} =$ ______.

Solution (20 points = 5+5+5+5)

(a) The pivots are in the first two columns of U, so x_3 and x_4 are the free variables. Setting $x_3 = 1, x_4 = 0$, we get (from the second row of $U\vec{x} = 0$) $x_2 = -1$

and (from the first row) $x_1 = 1 - 4x_2 = 5$; setting $x_3 = 0, x_4 = 1$, we get (from the second row) $x_2 = 3$ and (from the first row)$x_1 = -3 - 4x_2 = -15$. Hence, $N(A)$ is spanned by two special solutions as follows.

$$N(A) = x_3 \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -15 \\ 3 \\ 0 \\ 1 \end{pmatrix} \qquad \text{for all } x_3, x_4 \in \mathbb{R}.$$

(b) First, we need to find a particular solution. For this, we may set the free variables to $y_3 = y_4 = 0$. Thus, (from the second row of $U\vec{y} = b$) $y_2 = -6$ and (from the first row) $y_1 = 9 - 4y_2 = 33$. Hence, all the solution to the equations are given by the sum of the particular solution and any vector in the nullspace (all linear combinations of the special solutions):

$$\vec{y} = y_3 \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix} + y_4 \begin{pmatrix} -15 \\ 3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 33 \\ -6 \\ 0 \\ 0 \end{pmatrix} \qquad \text{for all } y_3, y_4 \in \mathbb{R}$$

(c) Nyarla first subtracted $\mathbf{2}$ times the first row from the second row, then subtracted $\mathbf{-1}$ time s the first row from the third row, then subtracted **3 times the second row from the third row**.

There are a couple of ways to solve this problem. The easiest is to remember that the L matrix, the product of the inverses of the elimination matrices, is simply composed of the multipliers for each of the elimination steps below each column. Under the first column of L we have 2 and -1, and these are thus the multiples of the first row that get subtracted from rows 2 and 3. Under the second column of L we have a 3, and this is the multiple of the second row that gets subtracted from the third row.

The other way to solve it is to just multiply L by U to get $A = LU$, and re-do the elimination process. Obviously, this is a bit more work, but is not too bad.

(d) Applying the same elimination operations in (c) to $A\vec{x}$ should give $U\vec{x}$. So, we have

$$\begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

Alternatively, we can just solve $U\vec{x}$ from $A\vec{x}$ as follows. Let $\vec{v} = U\vec{x}$. Then $L\vec{v} = UL\vec{x} = A\vec{x} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$. Thus, we can solve from the top as follows. $v_1 = 0$, $v_2 = 2 - 2v_1 = 2$, and $v_3 = 6 - 3v_2 + v_1 = 0$. Hence, $U\vec{x} = \vec{v} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

REMARK: Some students realized that $U\vec{x} = L^{-1}(A\vec{x})$. But several of these students did not get $L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -3 & 1 \end{pmatrix}$ correctly. Be careful that the inverse of $\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$ is *not* $\begin{pmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ -l_{31} & -l_{32} & 1 \end{pmatrix}$; the lower left entry should be $l_{21}l_{32} - l_{31}$. (Only for elimination matrices, which have nonzero entries below only a single diagonal, can you always invert just by flipping signs.) More generally, if you find yourself inverting a matrix, you should realize that there is probably an easier way to do it: to multiply $\vec{v} = L^{-1}(A\vec{x})$, it is easier to solve $L\vec{v} = A\vec{x}$ for $\vec{v}$ by elimination (especially since L is triangular, so you can just do forward substitution as above).

Problem 2: Which of the following (if any) are subspaces? For any that are *not* a subspace, give an example of how they violate a property of subspaces.

(I) Given some 3×5 matrix A with full row rank, the set of all solutions to $A\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(II) All vectors $\vec{x}$ with $\vec{x}^T\vec{y} = 0$ *and* $\vec{x}^T\vec{z} = 0$ for some given vectors $\vec{y}$ and $\vec{z}$.

(III) All 3×5 matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in their column space.

(IV) All 5×3 matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in their nullspace.

(V) All vectors $\vec{x}$ with $\|\vec{x} - \vec{y}\| = \|\vec{y}\|$ for some given fixed vector $\vec{y} \neq 0$.

Solution (20 points = 4+4+4+4+4)

(I) No. This is not a vector space because $\vec{x} = 0$ is not in this subspace.

(II) Yes. (This is actually just the left nullspace of the matrix whose columns are $\vec{y}$ and $\vec{z}$.)

(III) No. For example, the zero matrix is not in this subset.

(IV) Yes. If the nullspaces of A_1 and A_2 contain $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then any linear combination of these matrices does too:

$$(\alpha_1 A_1 + \alpha_2 A_2)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \alpha_1 A_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \alpha_2 A_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0; \text{ for all } \alpha_1, \alpha_2.$$

(V) No. For example, $2\vec{y}$ satisfies the condition (because $\|2\vec{y} - \vec{y}\| = \|\vec{y}\|$) but $\vec{y}$ does not satisfies the condition (because $\|\vec{y} - \vec{y}\| = 0 \neq \|\vec{y}\|$). This violates the fact that a subspace is preserved under multiplication by scalars.

REMARK: A common problem we saw in the grading is that some students do not know how to express a counterexample. A counterexample is simply a single specific element of the set that violates a specific property of subspaces, or a specific element that should be in the set but isn't (as in the case of the sets missing $\vec{0}$ above). One such example is all that is needed to disqualify a set as a subspace; no further abstract argument is necessary. If you were asked to find an "example" and you find yourself writing a long, abstract essay, you are probably making a mistake!

Problem 3: A is a matrix with a nullspace $N(A)$ spanned by the following three vectors:

$$\begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -1 \\ 3 \\ 1 \end{pmatrix}.$$

(α) Give a matrix B such that its column space $C(B)$ is the same as $N(A)$. (There is more than one correct answer.) [Thus, any vector $\vec{y}$ in the nullspace of A satisfies $B\vec{u} = \vec{y}$ for some $\vec{u}$.]

(β) Give a different possible answer to (α): another B with $C(B) = N(A)$.

(γ) For some vector $\vec{b}$, you are told that a particular solution to $A\vec{x} = \vec{b}$ is

$$\vec{x}_p = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Now, your classmate Zarkon tells you that a second solution is:

$$\vec{x}_Z = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix},$$

while your other classmate Hastur tells you "No, Zarkon's solution can't be right, but here's a second solution that is correct:"

$$\vec{x}_H = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix}.$$

Is Zarkon's solution correct, or Hastur's solution, or are both correct? (Hint: what should be true of $\vec{x} - \vec{x}_p$ if $\vec{x}$ is a valid solution?)

Solution (20 points = 5+5+10) (α) Since the nullspace is spanned by the given three vectors, we may simply take B to consist of the three vectors as columns, i.e.,

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ -1 & 1 & 3 \\ 3 & 4 & 1 \end{pmatrix}.$$

B need not be square (many students insisted on square solutions).

(β) For example, we may simply add a zero column to B:

$$B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & 0 \\ -1 & 1 & 3 & 0 \\ 3 & 4 & 1 & 0 \end{pmatrix}.$$

Or, we could interchange two columns. Or we could multiply one of the columns by -1. For example:

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -3 \\ 3 & 4 & -1 \end{pmatrix}.$$

Or we could replace one of the columns by a linear combination of that column with the other two columns (any invertible column operation). Or we could replace B by $-B$ or $2B$. There are many possible solutions. In any case, the solution shouldn't require any significant calculation!

(γ) Since any solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ is of the form $\vec{x}_p + \vec{n}$ for some vector $\vec{n}$ in the nullspace, the vector $\vec{x} - \vec{x}_p$ must lie in the nullspace $N(A)$. Thus, we want to look at:

$$\vec{x}_Z - \vec{x}_p = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -4 \end{pmatrix}, \quad \vec{x}_H - \vec{x}_p = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -3 \end{pmatrix}.$$

To determine whether a vector $\vec{y}$ lies in the nullspace $N(A)$, we can just check whether it is in the column space of B, i.e. check whether $B\vec{z} = \vec{y}$ has a solution. As we learned in class, we can check this just by doing elimination: if elimination produces a zero row in B, it should produce a zero row in the right-hand side. In terms of B from part (α) augmented by the right-hand side, this gives:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & -1 \\ -1 & 1 & 3 & 0 \\ 3 & 4 & 1 & a \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & 4 & a \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & a+4 \end{pmatrix}$$

We can get a solution if and only if $a = -4$. So Zarkon is correct.

REMARK: Several students apparently just stared at the nullspace vectors and found a linear combination that gave $\vec{x}_Z - \vec{x}_p$:

$$\vec{x}_Z - \vec{x}_p = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -4 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 4 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ -1 \\ 3 \\ 1 \end{pmatrix}.$$

Then they stared at Hastur's solution, couldn't find such a combination, and concluded that it was not a solution. This conclusion is correct *in this case*, and was awarded full marks because you were not asked to justify your solution. However, doing elimination is much more systematic and reliable, and ensures that there isn't a linear combination that you simply missed. Use elimination next time!

REMARK: Some students saw the zero components of $\vec{x}_Z - \vec{x}_p$, didn't see any corresponding zero components in the given nullspace vectors, and concluded that $\vec{x}_Z - \vec{x}_p$ was not in the nullspace. This is wrong: the key point is that $\vec{x}_Z - \vec{x}_p$ can be any vector in the nullspace, which means any linear combination of the given nullspace vectors. There are plenty of ways to combine nonzero vectors to get vectors with zero components!