18.06 Problem Set 9 Solution

Due Wednesday, 29 April 2009 at 4 pm in 2-106.

Total: 130 points.

Problem 1: Let $A = \begin{pmatrix} 1 & s \\ 1 & 3 \end{pmatrix}$, where s is some real number.

(a) Give a value of s where A is defective; use this s in the subsequent parts.

(b) Compute a set of eigenvectors and generalized eigenvectors (as defined in the handout) of A to give a complete basis for $\mathbb{R}^2$. (Use this basis in the subsequent parts.)

(c) For the column vector $\vec{u}_0 = (1,0)^{\mathrm{T}}$, compute $\vec{u}(t) = (I + e^{At})^{-1}\vec{u}_0$ (as an explicit formula with no matrix operations). (Hint: use the formula for $f(A)$ from the handout; note that $I = A^0$.)

(d) For the column vector $\vec{u}_0 = (1,0)^{\mathrm{T}}$, compute $\vec{u}_k = A^k\vec{u}_0$ (as an explicit formula with no matrix operations).

(e) Write an explicit formula for A^k, for any k (as an explicit formula with no matrix operations). (Consider your answer for the previous part, and ask what matrix you would multiply by an arbitrary vector to obtain A^k times that vector.)

(f) Suppose we perturb the matrix slightly, changing s to $s + 0.0001$. Does $\|\vec{u}_k\|$ grow more slowly or more quickly with k than when A was defective?

$\boxed{\text{Solution}}$ (30 points = 5+5+5+5+5+5)

(a) Since we need A to be defective, its two eigenvalues must be the same. $\det(A - \lambda I) = \lambda^2 - 4\lambda + (3 - s) = 0$. It has repeated roots when $3 - s = 4$, that is when $s = -1$. In this case, $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$.

(b) When $s = -1$, solving $\det(A - \lambda I) = \lambda^2 - 4\lambda + 4 = 0$ gives $\lambda = 2$.

$$A - \lambda I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \quad \Rightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

To find the generalized eigenvector, we solve the linear system

$$(A-\lambda I)\vec{v}_1' = \vec{v}_1 \quad \rightsquigarrow \quad \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \vec{v}_1' = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We can find a particular solution $\vec{v}_1' = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Then, we use Gram-Schmidt to make it perpendicular to $\vec{v}_1$ as follows.

$$\vec{v}_1^{(2)} = \vec{v}_1' - \frac{\vec{v}_1'^{\mathrm{T}}\vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

(c) First, we need to write $\vec{u}_0 = c_1\vec{v}_1 + c_2\vec{v}_1^{(2)}$.

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad c_1 = \frac{1}{2},\ c_2 = -1.$$

Let $f(X) = (1+e^{Xt})^{-1}$. Then $f'(X) = te^{Xt} \cdot (1+e^{Xt})^{-2}$. By the formula from the handout,

$$\begin{aligned}
\vec{u}(t) &= (I+e^{At})^{-1}\vec{u}_0 \\
&= c_1 f(\lambda)\vec{v}_1 + c_2\big(f(\lambda)\vec{v}_1^{(2)} + f'(\lambda)\vec{v}_1\big) \\
&= \frac{1}{2}\frac{1}{1+e^{2t}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \Big(\frac{1}{1+e^{2t}}\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} + \frac{te^{2t}}{(1+e^{2t})^2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\Big) \\
&= \frac{1}{1+e^{2t}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{te^{2t}}{(1+e^{2t})^2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\end{aligned}$$

REMARK: One may notice that the first term is exactly $\frac{1}{1+e^{2t}}\vec{u}_0$. This is because the eigenvalues of the two eigenvectors are the same. The second term is contributed by the generalized eigenvector $\vec{v}_1^{(2)}$, but it is a multiple of $\vec{v}_1$.

(d) Continuing with the calculation above, we have

$$\begin{aligned}
\vec{u}_k = A^k\vec{u}_0 &= c_1\lambda^k\vec{v}_1 + c_2\big(\lambda^k\vec{v}_1^{(2)} + k\lambda^{k-1}\vec{v}_1\big) \\
&= \frac{1}{2}2^k\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \Big(2^k\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} + k2^{k-1}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\Big) \\
&= 2^k\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k\cdot 2^{k-1}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2^{k-1}\begin{pmatrix} 2-k \\ k \end{pmatrix}.
\end{aligned}$$

(e) Since $A^k v_1 = \lambda^k v_1$ and $A^k v_1^{(2)} = \lambda^k v_1^{(2)} + k\lambda^{k-1} v_1$, we have

$$A^k \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2^k & -\frac{1}{2}2^k + k \cdot 2^{k-1} \\ -2^k & -\frac{1}{2}2^k - k \cdot 2^{k-1} \end{pmatrix} = 2^{k-1} \begin{pmatrix} 2 & k-1 \\ -2 & -k-1 \end{pmatrix}.$$

Hence,

$$\begin{aligned} A^k &= 2^{k-1} \begin{pmatrix} 2 & k-1 \\ -2 & -k-1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} \end{pmatrix}^{-1} = 2^{k-1} \begin{pmatrix} 2 & k-1 \\ -2 & -k-1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & -1 \end{pmatrix} \\ &= 2^{k-1} \begin{pmatrix} 2-k & -k \\ k & k+2 \end{pmatrix}. \end{aligned}$$

(f) If we change s to $s + 0.0001$, then to get the eigenvalue, we need to solve $\lambda^2 - 4\lambda + 4 - 0.0001 = 0$, that is $(\lambda - 2)^2 = 0.0001$. We get $\lambda_1 = 2.01$ and $\lambda_2 = 1.99$. Since $\lambda_1 > \lambda$, $\|\vec{u}_k\|$ is going to grow more quickly with k than when A was defective. More precisely, since an exponential always grows faster than any polynomial, $2.01^k = 1.05^k 2^k$ grows faster than $k2^k$.

Problem 2: True or false, with a good reason:

(a) A can't be similar to $-A$ unless $A = 0$.

(b) An invertible matrix can't be similar to a singular matrix.

(c) A symmetric matrix can't be similar to a nonsymmetric matrix.

(d) Any diagonalizable matrix is similar to a Hermitian matrix.

(e) If B is invertible, then AB and BA have the same eigenvalues.

$\boxed{\text{Solution}}$ (25 points = 5+5+5+5+5)

(a) False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is similar to $-A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ as follows.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -A.$$

Note that $-A$ flips the signs of the eigenvalues, but this doesn't mean it is not similar since the eigenvalues could come in positive and negative pairs.

Another counter-example would be any anti-symmetric $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It is similar to $-A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ as follows.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -A.$$

(b) True. All eigenvalues of an invertible matrix are nonzero, whereas a singular matrix has at least one eigenvalue 0. They cannot be similar to each other.

(c) False. For example,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

In fact, (even) if A is symmetric, MAM^{-1} is not generally symmetric unless M is orthogonal.

(d) False. For example, the matrix $A = (1+i)$. It is not similar to any Hermitian matrix. More generally, a Hermitian matrix always has real eigenvalues, whereas a diagonalizable matrix can have any type of eigenvalues.

(e) True. This is because $AB = B^{-1}(BA)B$ is similar to BA. They have the same eigenvalues.

Problem 3: This question concerns the second-order ODE $y'' + 10y' + 25y = 0$ with the initial conditions $y(0) = 2$, $y'(0) = 3$.

(a) Convert this into a matrix equation $d\vec{u}/dt = A\vec{u}$ by $u_1 = y$, $u_2 = y'$. The initial condition is $\vec{u}(0) =$ __________.

(b) Find the eigenvalues and eigenvectors of A. A is a __________ matrix.

(c) Find the solution $\vec{u}(t) = e^{At}\vec{u}(0)$, and hence the solution $y(t)$.

$\boxed{\text{Solution}}$ (15 points = 5+5+5)

(a) We can write

$$\begin{aligned} y' &= y' \\ y'' &= -25y - 10y' \end{aligned} \quad \Rightarrow \quad \frac{d\vec{u}}{dt} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -25 & -10 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}.$$

That is $A = \begin{pmatrix} 0 & 1 \\ -25 & -10 \end{pmatrix}$. The initial condition is $\vec{u}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

(b) Solving $\det(A - \lambda I) = \lambda^2 + 10\lambda + 25 = 0$ gives $\lambda = -5$, a double root.

$$A - \lambda I = \begin{pmatrix} 5 & 1 \\ -25 & -5 \end{pmatrix} \quad \Rightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -5 \end{pmatrix}.$$

So, A is a defective matrix.

(c) We need to find the second eigenvector by solving

$$(A - \lambda I)\vec{v}_1^{(2)} = \vec{v}_1 \quad \rightsquigarrow \quad \begin{pmatrix} 5 & 1 \\ -25 & -5 \end{pmatrix} \vec{v}_1^{(2)} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}.$$

We can find a particular solution $\vec{v}_1^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Here, we could have used the Gram-Schmidt to find the a solution that is perpendicular to $\vec{v}_1$. However, in order to find the solution $\vec{u}(t) = e^{At}\vec{u}(0)$, it suffices to use *any* particular solution. Besides the computation involving Gram-Schmidt tends to be complicated.

We need to write $\vec{u}(0)$ in terms of $\vec{v}_1$ and $\vec{v}_1^{(2)}$.

$$\begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \Rightarrow \quad c_1 = 2,\ c_2 = 13.$$

Thus, we have

$$\begin{aligned} \vec{u}(t) &= e^{At}\vec{u}_0 \\ &= c_1 e^{\lambda t}\vec{v}_1 + c_2\big(e^{\lambda t}\vec{v}_1^{(2)} + te^{\lambda t}\vec{v}_1\big) \\ &= 2e^{-5t}\begin{pmatrix} 1 \\ -5 \end{pmatrix} + 13\Big(e^{-5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + te^{-5t}\begin{pmatrix} 1 \\ -5 \end{pmatrix}\Big) \\ &= e^{-5t}\begin{pmatrix} 2 + 13t \\ 3 - 65t \end{pmatrix}. \end{aligned}$$

In particular, $y(t) = (13t + 2)e^{-5t}$.

Problem 4: Suppose that λ_1 is a double root of $\det(A - \lambda_1 I)$ for some A, but that $N(A - \lambda_1 I)$ is one dimensional, the span of a single eigenvector $\vec{x}_1$. A is thus defective. It turns out that $(A - \lambda_1 I)^2$ must always have a *two*-dimensional nullspace if λ_1 is a double root. Let $\vec{y}$ be a vector in the nullspace of $(A - \lambda_1 I)^2$ that is *not* in $N(A-\lambda_1 I)$. In what nullspace is $(A-\lambda_1 I)\vec{y}$? Hence, $(A-\lambda_1 I)\vec{y}$ is proportional to __________.

$\boxed{\text{Solution}}$ (10 points)

Since $\vec{y} \in N\big((A-\lambda_1 I)^2\big)$, $(A-\lambda_1 I)^2\vec{y} = 0$. Hence, $(A-\lambda_1 I)\big((A-\lambda_1 I)\vec{y}\big) = 0$. In another word, $(A-\lambda_1 I)\vec{y} \in N(A-\lambda_1 I)$ the nullspace of $A-\lambda_1 I$. Moreover, we know that $N(A-\lambda_1 I)$ is one dimensional, the span of a single eigenvector $\vec{x}_1$. This forces $(A-\lambda_1 I)\vec{y}$ to be proportional to $\vec{x}_1$ (and the ratio is nonzero, otherwise $\vec{y}$ itself would be in the nullspace of $A-\lambda_1 I$, leading to a contradiction.)

Problem 5: This is a Matlab problem using the SVD to perform image compression. This is not the best technique for image compression, but it showcases the SVD's ability to extract the important information from a matrix.

(a) Download `http://jdj.mit.edu/~stevenj/strang.jpg`, a grayscale image of a familiar fellow, to the directory that you launch Matlab from (e.g. your home directory on Athena). Each pixel is stored as a number from 0 (black) to 255 (white), so the image can be interpreted as a matrix A (in this case, a 404×303 matrix). Read it into Matlab and display it with the commands:

```
A = flipud(double(imread('strang.jpg')));
pcolor(A); shading interp; colormap('gray'); axis equal
```

(b) Now, compute the SVD $A = U\Sigma V^T$ using Matlab, create a new figure, and plot the distribution of singular values σ_i (the diagonals of Σ) on a log scale:

```
[U,S,V] = svd(A);
figure
semilogy(diag(S), 'o');
xlabel('index of singular value'); ylabel('singular values');
```

(c) Now, let's see what happens if we throw out all but the biggest 50 singular values, just setting the other ones to zero to make a new matrix `S2`:

```
S2 = S * diag([ones(1,50), zeros(1,size(S,2)-50)]);
figure
pcolor(U*S2*V'); shading interp; colormap('gray'); axis equal
```

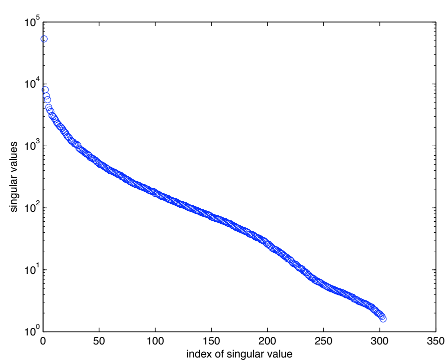
It should still look a lot like the original image: most of the information is in the biggest singular values and the corresponding singular vectors!

(d) Replace the two 50's in the previous commands to a smaller number, to keep fewer than 50 singular values. How small can you go before the image becomes unrecognizable? Which details of the image are the last to be blurred away?

$\boxed{\text{Solution}}$ (15 points)

```
>> A = flipud(double(imread('strang.jpg')));
>> pcolor(A); shading interp; colormap('gray'); axis equal
```

```
>> [U,S,V] = svd(A);
>> figure
>> semilogy(diag(S), 'o');
>> xlabel('index of singular value'); ylabel('singular values');
```



```
>> S2 = S * diag([ones(1,50), zeros(1,size(S,2)-50)]);
>> figure
>> pcolor(U*S2*V'); shading interp; colormap('gray'); axis equal
```

Next, we replace 50 by n =40, 35, 30, 25, 20, 15, 12, 10, 8, 6, 4 and print out the pictures.

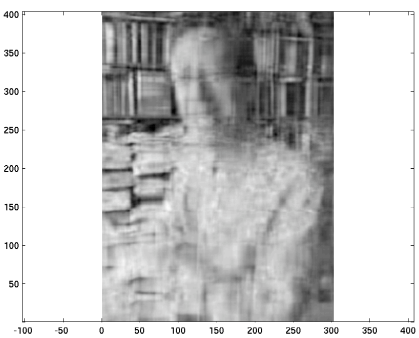
$n = 40$

$n = 35$

$n = 30$

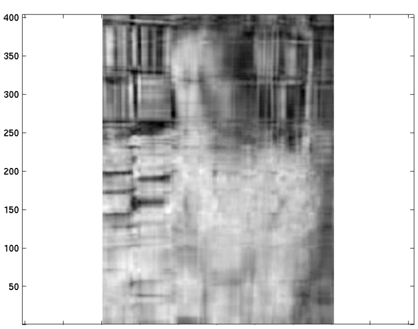
$n = 25$

$n = 20$

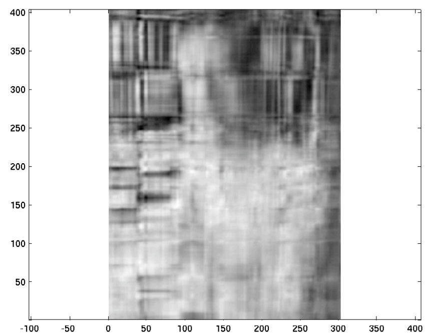


$n = 15$

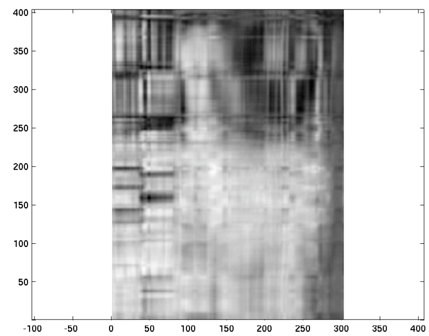
$n = 12$



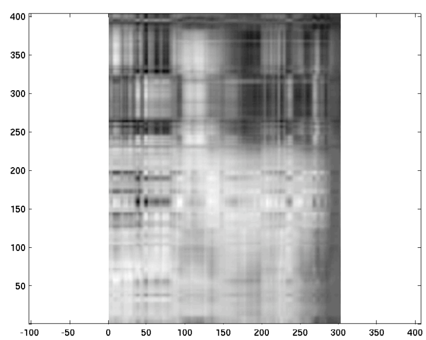
$n = 10$



$n = 8$



$n = 6$



$n = 4$

The singular values are dominated by the first 50 or, with the rest an order of magnitude or more smaller. This is an empirical observation (which is related to certain statistical properties), that for many images the singular values fall off roughly exponentially as shown in the graph, and is why the image can be roughly reconstructed from just a few of the largest singular values.

Problem 6: Find the eigenvalues and orthonormal eigenvectors of $A^{\mathrm{T}}A$ and AA^{T} for the Fibonacci matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Construct the singular value decomposition and verify that $A = U\Sigma V^{\mathrm{T}}$.

$\boxed{\text{Solution}}$ (15 points)

In this case, we are lucky that A is symmetric. So, the SVD is the same as the decomposition $S^{-1}\Sigma S$ if we take S to be orthogonal. But in order to illustrate the process of SVD, we write out the steps for treating general matrix A.

Step 1: Find the eigenvalues and orthonormal eigenvectors of $A^{\mathrm{T}}A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Solving $\det(A^{\mathrm{T}}A - \lambda I) = \lambda^2 - 3\lambda + 1$ gives $\lambda = \frac{1}{2}(3 \pm \sqrt{5})$.

$$\lambda_1 = \frac{3+\sqrt{5}}{2}, \quad A - \lambda_1 I = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{pmatrix}, \quad u_1' = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}.$$

We normalize it to get $u_1 = \frac{u_1'}{\|u_1'\|} = \left(\sqrt{\frac{5+\sqrt{5}}{2}}\right)^{-1} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} \approx \begin{pmatrix} 0.8507 \\ 0.5257 \end{pmatrix}$. To get the other (normalized eigenvector), one can simply change the sign before $\sqrt{5}$ and get $u_2 = \left(\sqrt{\frac{5-\sqrt{5}}{2}}\right)^{-1} \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix} \approx \begin{pmatrix} -0.5257 \\ 0.8507 \end{pmatrix}$.

Step 2: Find the eigenvalues and orthonormal eigenvectors of $AA^{\mathrm{T}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Since A is symmetric in our case, we will get the same answer, that is

$$v_1 = \left(\sqrt{\frac{5+\sqrt{5}}{2}}\right)^{-1} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} \approx \begin{pmatrix} 0.8507 \\ 0.5257 \end{pmatrix}$$
$$v_2 = \left(\sqrt{\frac{5-\sqrt{5}}{2}}\right)^{-1} \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix} \approx \begin{pmatrix} -0.5257 \\ 0.8507 \end{pmatrix}.$$

Step 3: The Singular Value Decomposition is

$$A = U\Sigma V^{\mathrm{T}} \approx \begin{pmatrix} 0.8507 & -0.5207 \\ 0.5257 & 0.8507 \end{pmatrix} \begin{pmatrix} 1.6180 & 0 \\ 0 & -0.6180 \end{pmatrix} \begin{pmatrix} 0.8507 & 0.5207 \\ -0.5257 & 0.8507 \end{pmatrix}$$

Here the singular values (the diagonal of Σ) are the square roots of the eigenvalues of $A^T A$ and AA^T, namely, $\sqrt{\frac{3+\sqrt{5}}{2}} = \frac{1+\sqrt{5}}{2}$ and $\sqrt{\frac{3-\sqrt{5}}{2}} = \frac{1-\sqrt{5}}{2}$.

Problem 7: If $A = QR$ with an orthogonal matrix Q (A is square), the SVD of A is almost the same as the SVD of R. Which of the three matrices U, Σ, and V must be different for A and R?

$\boxed{\text{Solution}}$ (10 points)

If $A = U\Sigma V^{\mathrm{T}}$ is the SVD for A, then $QR = U\Sigma V^{\mathrm{T}}$. Since Q is orthogonal, $Q^{-1} = Q^{\mathrm{T}}$ and hence $R = (Q^{\mathrm{T}}U)\Sigma V^{\mathrm{T}}$. Note that $(Q^{\mathrm{T}}U)^{\mathrm{T}}(Q^{\mathrm{T}}U) = U^{\mathrm{T}}QQ^{\mathrm{T}}U = U^{\mathrm{T}}U = I$. This implies that $Q^{\mathrm{T}}U$ is an orthogonal matrix. Hence, $R = (Q^{\mathrm{T}}U)\Sigma V^{\mathrm{T}}$ is an SVD for R.

REMARK: The condition that A is square ensures that Q is a square matrix in the QR factorization.

If A were not square, we would still have $Q^{\mathrm{T}}Q = I$. However, multiplying Q^{T} by both sides of the SVD, but then you would get $Q^{\mathrm{T}}U$, which is not an square matrix and hence not orthogonal and hence not the SVD (at least in the form learned in class).

Problem 8: Suppose $\vec{u}_1, \ldots, \vec{u}_n$ and $\vec{v}_1, \ldots, \vec{v}_n$ are two orthonormal bases for $\mathbb{R}^n$. Write a formula for the matrix A that transforms each $\vec{v}_j$ into $\vec{u}_j$ to give $A\vec{v}_1 = \vec{u}_1$, ..., $A\vec{v}_n = \vec{u}_n$. A is a/an ____________________ matrix (hint: check $A^{\mathrm{T}}A$).

$\boxed{\text{Solution}}$ (10 points)

If we let U be the matrix whose columns are $\vec{u}_1, \ldots \vec{u}_n$ and Let V be the matrix whose columns are $\vec{v}_1, \ldots, \vec{v}_n$. Then, the condition $A\vec{v}_1 = \vec{u}_1$, ..., $A\vec{v}_n = \vec{u}_n$ says exactly $AV = U$. Hence $A = UV^{-1} = UV^{\mathrm{T}}$. A is an orthogonal matrix because $A^{\mathrm{T}}A = (UV^{\mathrm{T}})^{\mathrm{T}}UV^{\mathrm{T}} = VU^{\mathrm{T}}UV^{\mathrm{T}} = VV^{\mathrm{T}} = I$.

Note also that $A = UIV^{\mathrm{T}}$ is the SVD for A, where the singular value matrix $\Sigma = I$.