18.06 Problem Set 8

Due Wednesday, 22 April 2009 at 4pm in 2-106.

1. If A is real-symmetric, it has real eigenvalues. What can you say about the eigenvalues if A is real and anti-symmetric ($A = -A^T$)? Give both a general explanation for any $n \times n$ A (similar to what we did in class and in the book) and check by finding the eigenvalues a 2×2 anti-symmetric example matrix.

2. Find an orthogonal matrix Q that diagonalizes $A = \begin{pmatrix} -2 & 6 \\ 6 & 7 \end{pmatrix}$, i.e. so that $Q^T A Q = \Lambda$ where Λ is diagonal. What is Λ?

3. Even if the real matrix A is rectangular, the block matrix $B = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$ is symmetric. An eigenvector $\mathbf{x}$ of B satisfies $B\mathbf{x} = \lambda \mathbf{x}$ with:

$$\mathbf{x} = \begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix}, \qquad \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix},$$

and thus $A\mathbf{z} = \lambda \mathbf{y}$ and $A^T \mathbf{y} = \lambda \mathbf{z}$.

   (a) Show that $-\lambda$ is also an eigenvalue of B, with the eigenvector $(\mathbf{y}, -\mathbf{z})$.
   (b) Show that $A^T A \mathbf{z} = \lambda^2 \mathbf{z}$, so that λ^2 is an eigenvalue of $A^T A$.
   (c) Show that λ^2 is also an eigenvalue of AA^T by finding a corresponding eigenvector.
   (d) If $A = I$ (2×2), find all four eigenvalues and eigenvectors of B.

4. True or false (give a reason if true, or a counter-example if false).

   (a) A matrix with real eigenvalues and real eigenvectors is symmetric.
   (b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.
   (c) The inverse of a symmetric matrix is symmetric.
   (d) The eigenvector matrix S of a symmetrix matrix is symmetric.
   (e) A complex symmetric matrix has real eigenvalues.
   (f) If A is symmetric, then e^{iA} is symmetric.
   (g) If A is Hermitian, then e^{iA} is Hermitian.

5. For which s is A positive definite?

$$A = \begin{pmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{pmatrix}.$$

6. If A has full column rank, and C is positive-definite, show that $A^T CA$ is positive definite. (Recall that $A^T CA$ is an important matrix; for example, it arose in lecture 13 on graphs and networks, section 8.2 of the text.)

7. For $f_1(x,y) = x^4/4 + x^2 + x^2 y + y^2$ and $f_2(x,y) = x^3 + xy - x$, find the second-derivative matrices H_1 and H_2, where:
$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}.$$
Find the minimum point of f_1 (and check that H_1 is positive-definite there). Find the saddle point of f_2 (look only where the first derivatives are zero, and check that H_2 has two eigenvalues with opposite signs).

8. (a) Give an explicit formula for $\mathbf{u}_k = A^k \mathbf{u}_0$, where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{u}_0 = \begin{pmatrix} 1 & 2 \end{pmatrix}^T$.
   (b) Although you should find that A's eigenvalues and eigenvectors are not real, give explicit values for $\mathbf{u}_{100}$, $\mathbf{u}_{101}$, $\mathbf{u}_{102}$, $\mathbf{u}_{103}$, showing that your formula gives real results.
   (c) $\mathbf{u}_{k+n} = \mathbf{u}_k$ for what value(s) of n?

9. For what (real) values of s does $d\mathbf{u}/dt = A\mathbf{u}$ have exponentially growing solutions, where
$$A = \begin{pmatrix} -1 & s \\ 2 & -3 \end{pmatrix}?$$