18.06 Problem Set 8 Solution

Due Wednesday, 22 April 2009 at 4 pm in 2-106.

Total: 160 points.

Problem 1: If A is real-symmetric, it has real eigenvalues. What can you say about the eigenvalues if A is real and anti-symmetric ($A = -A^{\mathrm{T}}$)? Give both a general explanation for any $n \times n$ A (similar to what we did in class and in the book) and check by finding the eigenvalues a 2×2 anti-symmetric example matrix.

$\boxed{\text{Solution}}$ (15 points = 10(proof) + 5(example))

If λ is an eigenvalue of A with a nonzero eigenvector v, that is $Av = \lambda v$. Then, on one hand, we have $v^{\mathrm{H}}Av = v^{H}\lambda v = \lambda\|v\|^2$, and on the other hand,

$$v^{\mathrm{H}}Av = (-v^{\mathrm{H}}A^{\mathrm{T}})v = -(Av)^{\mathrm{H}}v = -(\lambda v)^{\mathrm{H}}v = -\bar{\lambda}\|v\|^2.$$

Since v is nonzero, $\|v\|^2 > 0$. We conclude that $\lambda = -\bar{\lambda}$. This implies that λ is purely imaginary, that is the real part of λ is zero.

For example, we take $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. (Since A is anti-symmetric, its diagonal entries must be zero.) We then solve $\det(A-\lambda I) = \lambda^2+1 = 0$ to get $\lambda_1 = i, \lambda_2 = -i$. They are purely imaginary numbers.

Problem 2: Find an orthogonal matrix Q that diagonalizes $A = \begin{pmatrix} -2 & 6 \\ 6 & 7 \end{pmatrix}$, i.e. so that $Q^T AQ = \Lambda$ where Λ is diagonal. What is Λ?

$\boxed{\text{Solution}}$ (10 points)

Since A is real-symmetric, we should be able to get orthonormal eigenvectors, and then Q is just the matrix whose columns are the eigenvectors (as in class and the textbook), and Λ is the diagonal matrix of eigenvalues. So, we just solve for the eigenvalues and eigenvectors of A. To get the eigenvalues, we solve $\det(A - \lambda I) = 0 = \lambda^2 - 5\lambda - 50$, obtaining $\lambda_1 = 10$ and $\lambda_2 = -5$. Since the eigenvalues are distinct, the eigenvectors are automatically orthogonal, and we just need to normalize them to have length 1:

$$\lambda_1 = 10, \quad A - \lambda_1 I = \begin{pmatrix} -12 & 6 \\ 6 & -3 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad q_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

$$\lambda_2 = -5, \quad A - \lambda_2 I = \begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad q_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}.$$

Hence, we have

$$Q = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 10 & 0 \\ 0 & -5 \end{pmatrix}$$

Problem 3: Even if the real matrix A is rectangular, the block matrix $B = \begin{pmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{pmatrix}$ is symmetric. An eigenvector $\vec{x}$ of B satisfies $B\vec{x} = \lambda\vec{x}$ with:

$$\vec{x} = \begin{pmatrix} \vec{y} \\ \vec{z} \end{pmatrix}, \qquad \begin{pmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} \vec{y} \\ \vec{z} \end{pmatrix} = \lambda \begin{pmatrix} \vec{y} \\ \vec{z} \end{pmatrix},$$

and thus $A\vec{z} = \lambda\vec{y}$ and $A^{\mathrm{T}}\vec{y} = \lambda\vec{z}$.

(a) Show that $-\lambda$ is also an eigenvalue of B, with the eigenvector $(\vec{y}, -\vec{z})^{\mathrm{T}}$.

(b) Show that $A^{\mathrm{T}}A\vec{z} = \lambda^2\vec{z}$, so that λ^2 is an eigenvalue of $A^{\mathrm{T}}A$.

(c) Show that λ^2 is also an eigenvalue of AA^{T} by finding a corresponding eigenvector.

(d) If $A = I$ (2×2), find all four eigenvalues and eigenvectors of B.

$\boxed{\text{Solution}}$ (25 points = 5+5+5+10)

(a) We check this by direct computation.

$$B \begin{pmatrix} \vec{y} \\ -\vec{z} \end{pmatrix} = \begin{pmatrix} -A\vec{z} \\ A^{\mathrm{T}}\vec{y} \end{pmatrix} = \begin{pmatrix} -\lambda\vec{y} \\ \lambda\vec{z} \end{pmatrix} = -\lambda \begin{pmatrix} \vec{y} \\ -\vec{z} \end{pmatrix}.$$

Hence $-\lambda$ is also an eigenvalue of B, with the eigenvector $\begin{pmatrix} \vec{y} \\ -\vec{z} \end{pmatrix}$.

(b) Again, we check by direct computation.

$$A^{\mathrm{T}}A\vec{z} = A^{\mathrm{T}}(\lambda\vec{y}) = \lambda A^{\mathrm{T}}\vec{y} = \lambda(\lambda\vec{z}) = \lambda^2\vec{z}.$$

Hence, λ^2 is an eigenvalue of $A^{\mathrm{T}}A$ with eigenvector $\vec{z}$.

(c) By "symmetry", it is not hard to guess that $\vec{y}$ may be an eigenvector of AA^{T}. Indeed,

$$AA^{\mathrm{T}}\vec{y} = A(\lambda\vec{z}) = \lambda A\vec{z} = \lambda(\lambda\vec{y}) = \lambda^2\vec{y}.$$

Hence, λ^2 is an eigenvalue of AA^{T} with eigenvector $\vec{y}$.

(d) We can use the results from part (c) to solve this quickly. If $A = I$, then $A^{\mathrm{T}}A = I$ has an eigenvalue λ^2. But the eigenvalues of I are 1, so $\lambda^2 = 1$, so λ must be 1 or -1. But from (a), if 1 is an eigenvalue then -1 must also be an eigenvalue and vice versa, so the eigenvalues are 1 and -1. Furthermore, from (c), $\vec{y}$ is given by the eigenvectors of $A^{\mathrm{T}}A = I$, which are just $(1,0)^{\mathrm{T}}$ and $(0,1)^{\mathrm{T}}$. Furthermore, since we were given that $A^{\mathrm{T}}\vec{y} = \lambda\vec{z}$, it follows that $\vec{z} = \pm\vec{y}$ for $\lambda = \pm 1$. Hence, the eigenvectors of $\lambda = 1$ are:

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

and the eigenvectors of $\lambda = -1$ are:

$$\vec{x}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{x}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

(There are also more laborious ways to get the same result. For example, once we know that $\lambda = \pm 1$, we could compute the nullspace of $B \pm I$.)

Problem 4: True or false (give a reason if true, or a counter-example if false).

(a) A matrix with real eigenvalues and real eigenvectors is symmetric.

(b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.

(c) The inverse of a symmetric matrix is symmetric.

(d) The eigenvector matrix S of a symmetrix matrix is symmetric.

(e) A complex symmetric matrix has real eigenvalues.

(f) If A is symmetric, then e^{iA} is symmetric.

(g) If A is Hermitian, then e^{iA} is Hermitian.

$\boxed{\text{Solution}}$ (35 points = 5+5+5+5+5+5+5)

(a) False. For example, $A = \begin{pmatrix} 2 & 2 \\ 0 & 3 \end{pmatrix}$ has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ with eigenvectors $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, respectively.

REMARK: For an upper triangular matrix A, the eigenvalues are exactly the elements on the diagonal.

(b) False. For example, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

REMARK: True is **acceptable** if the students assume that A is diagonalizable, since we had handled only diagonalizable matrices up to that point in the course. In this case, we may normalize the eigenvectors so that they form an orthonormal basis. Now, we write $A = S^{-1}\Lambda S$. Since the eigenvectors form an orthonormal basis, $S^{-1} = S^{\mathrm{T}}$. Hence,

$$A^{\mathrm{T}} = (S^{\mathrm{T}}\Lambda S)^{\mathrm{T}} = S^{\mathrm{T}}\Lambda^{\mathrm{T}}(S^{\mathrm{T}})^{\mathrm{T}} = S^{\mathrm{T}}\Lambda S = A$$

is symmetric.

(c) True. If A is a symmetric matrix, $(A^{-1})^{\mathrm{T}}A = (A^{-1})^{\mathrm{T}}A^{\mathrm{T}} = (AA^{-1})^{\mathrm{T}} = I^{\mathrm{T}} = I$; this says $(A^{-1})^{\mathrm{T}}$ is also the inverse of A, which is of course the same as A^{-1}. Hence A^{-1} is symmetric.

(d) False. For example, $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Then $\det(A - \lambda I) = \lambda^2 - 2\lambda - 3 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = 3$. We solve the eigenvectors to be $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively. So, $S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Another way to see this is false is that one can freely multiply any column of S by a nonzero constant, but this will not preserve the symmetry.

(e) False. For example, $A = (i)$, the 1×1 matrix. The eigenvalue is i; it is not a real number.

(f) True. Indeed, $(e^{iA})^{\mathrm{T}} = e^{(iA)^{\mathrm{T}}} = e^{iA}$.

(g) False. $(e^{iA})^{\mathrm{H}} = e^{(iA)^{\mathrm{H}}} = e^{-iA^{\mathrm{H}}} = e^{-iA}$. It is typically not the same as e^{iA}. Taking $A = (1)$, the 1×1 matrix, would be a good enough example because $e^{iA} = (e^i)$, which is not a real number.

Problem 5: For which s is A positive definite?

$$A = \begin{pmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{pmatrix}.$$

$\boxed{\text{Solution}}$ (15 points)

Method 1: We do elimination and check the pivots are all positive numbers.

$$A = \begin{pmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{pmatrix} \rightsquigarrow \begin{pmatrix} s & -4 & -4 \\ 0 & s - \frac{16}{s} & -4 - \frac{16}{s} \\ 0 & -4 - \frac{16}{s} & s - \frac{16}{s} \end{pmatrix}$$
$$\rightsquigarrow \begin{pmatrix} s & -4 & -4 \\ 0 & s - \frac{16}{s} & -4 - \frac{16}{s} \\ 0 & -4 - \frac{16}{s} & s - \frac{16}{s} \end{pmatrix} \rightsquigarrow \begin{pmatrix} s & -4 & -4 \\ 0 & s - \frac{16}{s} & -4 - \frac{16}{s} \\ 0 & 0 & \frac{s^2-16}{s} - (-4 - \frac{16}{s})^2/(s - \frac{16}{s}) \end{pmatrix}$$

We need all pivots to be bigger than 0.

The first pivot is s, which gives $s > 0$. The second pivot is $s - \frac{16}{s}$, which gives $\frac{s^2-16}{s} > 0$. Since we already have $s > 0$, $s^2 > 16$ and hence $s > 4$. Now, the third pivot > 0 gives

$$\begin{aligned} & (s - 16/s)^2 > (-4 - 16/s)^2 \\ \Rightarrow\ & (s^2 - 16)^2 > (4s + 16)^2 \\ \Rightarrow\ & s^4 - 32s^2 + 256 > 16s^2 + 128s + 256 \\ \Rightarrow\ & s^4 - 48s^2 - 128s > 0 \\ \Rightarrow\ & s^3 - 48s - 128 > 0 \\ \Rightarrow\ & (s+4)^2(s-8) > 0 \end{aligned}$$

Hence $s > 8$.

Method 2: Since A is symmetric, it is diagonalizable. We just need to make all the eigenvalues of A positive.

Consider

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} s-\lambda & -4 & -4 \\ -4 & s-\lambda & -4 \\ -4 & -4 & s-\lambda \end{pmatrix} \\ &= (s-\lambda)^3 + 2 \cdot (-4)^3 - 3 \cdot (-4)^2(s-\lambda) \\ &= (s-\lambda)^3 - 48(s-\lambda) - 128 = (s-\lambda+4)^2(s-\lambda-8) \end{aligned}$$

Hence the eigenvalues are $\lambda_1 = \lambda_2 = s + 4$ and $\lambda_3 = s - 8$. If A is positive definite, $s + 4 > 0, s - 8 > 0 \Rightarrow s > 8$.

REMARK: The factorization does not come out from nowhere. One way to see it is that we know beforehand, when $s - \lambda = -4$, $A - \lambda I = \begin{pmatrix} -4 & -4 & -4 \\ -4 & -4 & -4 \\ -4 & -4 & -4 \end{pmatrix}$ is of rank 1 and hence there should be a 2-dimensional nullspace. So, we are expecting two eigenvalues $s + 4$. The $s - \lambda - 8$ term might be harder to guess. But knowing two roots of the equation, one can get the third by high-school synthetic division. (One could also look up the formula for cubic equations on Wikipedia, but that's a lot more messy!)

REMARK: There is another criterion by taking the determinants of the main diagonal "submatrices", which we did not cover in the class. It says that a symmetric matrix A is positive definite if and only if the determinant of upper left $r \times r$ block is > 0 for any r (including $r = n$, which corresponds to the determinant of A).

Problem 6: If A has full column rank, and C is positive-definite, show that $A^{\mathrm{T}}CA$ is positive definite. (Recall that $A^{\mathrm{T}}CA$ is an important matrix; for example, it arose in lecture 13 on graphs and networks, section 8.2 of the text.)

$\boxed{\text{Solution}}$ (10 points)

Since C is positive-definite, $y^{\mathrm{T}}Cy > 0$ for any $y \neq 0$ in $\mathbb{R}^n$. Now, we need to show that $z^{\mathrm{T}}A^{\mathrm{T}}CAz > 0$ for any $z \neq 0$ in $\mathbb{R}^n$. We can rewrite it as $z^{\mathrm{T}}A^{\mathrm{T}}CAz = (Az)^{\mathrm{T}}C(Az)$. Since A has full column rank, $N(A) = \{0\}$ and in particular, $Az \neq 0$ in $\mathbb{R}^n$. Therefore, we have $(Az)^{\mathrm{T}}C(Az) > 0$. This implies that $A^{\mathrm{T}}CA$ is positive definite.

Problem 7: For $f_1(x, y) = x^4/4 + x^2 + x^2y + y^2$ and $f_2(x, y) = x^3 + xy - x$, find the second-derivative matrices H_1 and H_2, where:

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}.$$

Find the minimum point of f_1 (and check that H_1 is positive-definite there). Find the saddle point of f_2 (look only where the first derivatives are zero, and check that H_2 has two eigenvalues with opposite signs).

$\boxed{\text{Solution}}$ (20 points = 10+10)

For $f_1(x,y)$, we first solve for the stationary point

$$\frac{\partial f_1}{\partial x} = x^3 + 2x + 2xy = 0, \tag{1}$$
$$\frac{\partial f_1}{\partial y} = x^2 + 2y = 0 \tag{2}$$

From (2), we have $y = -x^2/2$. Plug this into (1), we have $2x = 0$ and hence the only critical point is $x = y = 0$. At this point,

$$H_1 = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 3x^3 + 2 + 2y & 2x \\ 2x & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

It is positive definite and hence $(0,0)$ is a minimal point of $f_1(x,y)$.

REMARK: The problem for $f_1 = x^4/4 + x^2y + y^2$ as originally stated, you get a curve of minima $x^2 + 2y = 0$, and H_1 is only positive semidefinite.

For $f_2(x,y)$, we first solve for the stationary point

$$\frac{\partial f_2}{\partial x} = 3x^2 + y - 1 = 0,$$
$$\frac{\partial f_2}{\partial y} = x = 0$$

This implies that $y = 1$. At this point $(0,1)$,

$$H_2 = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 6x & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of H_2 at $(0,1)$ is the solution to $\det(H_2 - \lambda I) = \lambda^2 - 1$, which are $\lambda_1 = 1$ and $\lambda_2 = -1$. They are with opposite signs and hence $(0,1)$ is a saddle point of $f_2(x,y)$.

Problem 8:

(a) Give an explicit formula for $\vec{u}_k = A^k \vec{u}_0$, where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\vec{u}_0 = \begin{pmatrix} 1 & 2 \end{pmatrix}^{\mathrm{T}}$.

(b) Although you should find that A's eigenvalues and eigenvectors are not real, give explicit values for $\vec{u}_{100}$, $\vec{u}_{101}$, $\vec{u}_{102}$, $\vec{u}_{103}$, showing that your formula gives real results.

(c) $\vec{u}_{k+n} = \vec{u}_k$ for what value(s) of n?

$\boxed{\text{Solution}}$ (20 points = 10+5+5)

(a) Solving $\det(A - \lambda I) = \lambda^2 + 1 = 0$ gives $\lambda_1 = i$ and $\lambda_2 = -i$.

$$\lambda_1 = i, \quad A - \lambda_1 I = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\lambda_2 = -i, \quad A - \lambda_2 I = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Then, we solve

$$\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1-2i}{2} \\ \frac{1+2i}{2} \end{pmatrix}$$

Hence,

$$\vec{u}_k = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 = \frac{1-2i}{2} \cdot i^k \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1+2i}{2} \cdot (-i)^k \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

REMARK: As said in class, when A is a real matrix, having a complex eigenvalue λ with eigenvector v, then $\bar{\lambda}$ is also an eigenvalue and $\bar{v}$ is one of the eigenvectors. Moreover, when we try to solve for c_1 and c_2, we have $c_1 = \bar{c}_2$. One can further simplify the result we have to be

$$\vec{u}_k = 2\mathrm{Re}\left(\frac{1-2i}{2} \cdot i^k \begin{pmatrix} 1 \\ i \end{pmatrix} \right) = \begin{pmatrix} \mathrm{Re}\big((1-2i)i^k\big) \\ \mathrm{Re}\big((1-2i)i^{k+1}\big) \end{pmatrix}$$

(b) We have

$$\vec{u}_{100} = \frac{1-2i}{2} \cdot 1 \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1+2i}{2} \cdot 1 \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$$\vec{u}_{101} = \frac{1-2i}{2} \cdot i \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1+2i}{2} \cdot (-i) \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

$$\vec{u}_{102} = \frac{1-2i}{2} \cdot 1 \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1+2i}{2} \cdot (-1) \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

$$\vec{u}_{103} = \frac{1-2i}{2} \cdot (-i) \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1+2i}{2} \cdot i \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

A faster way might be to use $\vec{u}_{k+1} = A\vec{u}_k$ to compute $\vec{u}_{101}, \vec{u}_{102}, \vec{u}_{103}$ after we computed $\vec{u}_{100}$.

(c) Since $i^{k+4} = i^k$ and $(-i)^{k+4} = (-i)^k$, We have $\vec{u}_{k+4} = \vec{u}_k$. From this, we can of course say $\vec{u}_{k+4m} = \vec{u}_k$ for any m.

Another way to see this is to check first several terms and find out that $\vec{u}_5 = \vec{u}_1$. Then we know $\vec{u}_{k+4} = A^{k-1}\vec{u}_5 = A^{k-1}\vec{u}_1 = \vec{u}_k$.

Problem 9: For what (real) values of s does $d\vec{u}/dt = A\vec{u}$ have exponentially growing solutions, where

$$A = \begin{pmatrix} -1 & s \\ 2 & -3 \end{pmatrix}?$$

$\boxed{\text{Solution}}$ (10 points)

We need only to find the eigenvalues of A. Solving $\det(A - \lambda I) = \lambda^2 + 4\lambda + 3 - 2s$ would give the two eigenvalues of A, $\lambda = \frac{-4 \pm \sqrt{4+8s}}{2}$. The system has exponentially growing solution if and only if one of the solution has a solution with positive real part. If $4 + 8s < 0$, i.e., the eigenvalues are complex, the real part is equal to -2, which will not give exponentially growing solution. Thus, $4 + 8s \geq 0$, i.e, we have two real eigenvalues. The bigger one should be greater than 0 in order to get the exponentially growing solution.

$$\frac{-4 \pm \sqrt{4+8s}}{2} > 0 \quad \Rightarrow \quad 4 + 8s > 4^2 \quad \Rightarrow \quad s > 3/2.$$