18.06 Problem Set 7

Due Wednesday, 15 April 2009 at 4pm in 2-106.

1. Diagonalize $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and compute $S\Lambda^k S^{-1}$ to prove this formula for A^k:

$$A^k = \frac{1}{2}\begin{pmatrix} 3^k+1 & 3^k-1 \\ 3^k-1 & 3^k+1 \end{pmatrix}$$

2. Consider the sequence of numbers $f_0, f_1, f_2, \ldots$, defined by the recurrence relation $f_{n+2} = 2f_{n+1} + 2f_n$, starting with $f_0 = f_1 = 1$ (giving 1, 1, 4, 10, 28, 76, 208, ...).

   (a) As we did for the Fibonacci numbers in class (and in the book), express this process as repeated multiplication of a vector $\mathbf{u}_k = (f_{k+1}, f_k)^T$ by a matrix A: $\mathbf{u}_{k+1} = A\mathbf{u}_k$, and thus $\mathbf{u}_k = A^k\mathbf{u}_0$. What is A?

   (b) Find the eigenvalues of A, and thus explain that the ratio f_{k+1}/f_k tends towards __________ as $k \to \infty$. Check this by computing f_{k+1}/f_k for the first few terms in the sequence.

   (c) Give an explicit formula for f_k (it can involve powers of numbers, but not powers of matrices) by expanding f_0 in the basis of the eigenvectors of A.

   (d) If we apply the recurrence relation in *reverse*, we use the formula: $f_n = f_{n+2}/2 - f_{n+1}$ (just solving the previous recurrence formula for f_n). Show that you get the *same* reverse formula if you just compute A^{-1}.

   (e) What does $|f_k/f_{k+1}|$ tend towards as $k \to -\infty$ (i.e. after we apply the formula in reverse many times)? (Very little calculation required!)

3. Suppose that $A = S\Lambda S^{-1}$. Take determinants to prove that $\det A$ is the product of the eigenvalues of A. (This quick proof only works when A is ____________________.)

4. In this problem, you will show that the trace of a matrix (the sum of the diagonal entries) is equal to the sum of the eigenvalues, by first showing that AB and BA have the *same trace* for any matrices A and B. Follow the following steps:

   (a) The explicit formula for the entries c_{ij} of $C = AB$ is $c_{ij} = \sum_k a_{ik}b_{kj}$ (where a_{ik} and b_{kj} are the entries of A and B, respectively). The trace of C is $\sum_i c_{ii}$. Write down the explicit formula for the entries d_{ij} of the product $D = BA$. By plugging these matrix-multiply formulas into the formulas for the trace of $C = AB$ and $D = BA$, and comparing them, prove that AB and BA have the same trace.

   (b) $A = S\Lambda S^{-1}$, assuming A is ____________________. Combining this factorization with the fact you proved in (a), show that the trace of A is the same as the trace of Λ, which is sum of the eigenvalues.

5. Suppose $A^2 = A$. (This does *not* mean $A = I$, since A might not be invertible; it might be a projection onto a subspace, for example.)

   (a) Explain why any eigenvector with $\lambda = 0$ is in the ________________ space of A, *and vice versa* (any nonzero vector in that space is an eigenvector with $\lambda = 0$).

   (b) Explain why any eigenvector with $\lambda = 1$ is in the ________________ space of A, *and vice versa* (any nonzero vector in that space is an eigenvector with $\lambda = 1$). (Hint: first explain why each column of A is an eigenvector.)

   (c) Conclude from the dimensions of these subspaces that any such A must have a full set of independent eigenvectors and hence be diagonalizable.

6. A genderless alien society survives by cloning/budding. Every year, 5% of young aliens become old, 3% of old aliens become dead, and 1% of the old aliens and 2% of the dead aliens are cloned into new young aliens. The population can be described by a Markov process:

$$\begin{pmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{pmatrix}_{\text{year } k+1} = A \begin{pmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{pmatrix}_{\text{year } k}$$

(a) Give the Markov matrix A, and compute (without Matlab) the steady-state young/old/dead population fractions.

(b) In Matlab, enter your matrix A and a random starting vector `x = rand(3,1); x = x / sum(x)` (normalized to sum to 1). Now, compute the population for the first 100 years, and plot it versus time, by the following Matlab code:

```
p = [];
for k = 0:99
   p = [ p, A^k * x ];
end
plot([0:99], p')
legend('young', 'old', 'dead')
xlabel('year'); ylabel('population fraction');
```

Check that the final population `p(:,end)` is close to your predicted steady state.

(c) In Matlab, compute A to a large power A^{1000} (in Matlab: `A^1000`). Explain why you get what you do, in light of your answer to (a).

7. If A is *both* a symmetric matrix and a Markov matrix, why is its steady-state eigenvector $(1,1,\ldots,1)^T$?

8. Find the λ's and $\mathbf{x}$'s so that $\mathbf{u} = e^{\lambda t}\mathbf{x}$ is a solution of

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} \mathbf{u}. \tag{1}$$

Make a linear combination of these solutions to solve this equation with the initial condition $\mathbf{u}(0) = (5,-2)^T$.

9. Explain how to write an equation $\alpha \frac{d^2y}{dt^2} + \beta \frac{dy}{dt} + \gamma y = 0$ as a vector equation $M \frac{d\mathbf{u}}{dt} = A\mathbf{u}$.

10. A matrix A is antisymmetric, or "skew" symmetric, which means that $A^T = -A$. Prove that the matrix $Q = e^{At}$ is orthogonal: transpose the series for $Q = e^{At}$ to show that you get the series for e^{-At}, and thus $Q^T Q = I$. Therefore, if $\mathbf{u}(t) = e^{At}\mathbf{u}(0)$ is *any* solution to the system $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$, then we know that $\|\mathbf{u}(t)\| / \|\mathbf{u}(0)\| =$ __________.

11. If $A^2 = A$, show from the infinite series that $e^{At} = I + (e^t - 1)A$. For $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, this gives $e^{At} =$ __________.

12. Assume A is diagonalizable with real eigenvalues. What condition on the eigenvalues of A ensures that the solutions of $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ will *not* blow up for $t \to \infty$? In comparison, what condition on the eigenvalues of A ensures that solutions of the linear recurrence relation $\mathbf{u}_{k+1} = A\mathbf{u}_k$ will *not* blow up for $k \to \infty$?