18.06 Problem Set 7 Solution

Due Wednesday, 15 April 2009 at 4 pm in 2-106.
Total: 150 points.

Problem 1: Diagonalize $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and compute $S\Lambda^k S^{-1}$ to prove this formula for A^k:

$$A^k = \frac{1}{2}\begin{pmatrix} 3^k+1 & 3^k-1 \\ 3^k-1 & 3^k+1 \end{pmatrix}.$$

Solution (15 points)

Step 1: eigenvalues. $\det(A-\lambda I) = \lambda^2 - \text{trace}(A) + \det(A) = \lambda^2 - 4\lambda + 3 = 0$. The solutions are $\lambda_1 = 1, \lambda_2 = 3$.

Step 2: solve for eigenvectors.

$$\lambda_1 = 1, \quad A - \lambda I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\lambda_2 = 3, \quad A - \lambda I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Step 3: let $S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ be the matrix whose columns are eigenvectors. Let $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ be the matrix for eigenvalues. Then we have

$$A^k = S\Lambda^k S^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^k\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3^k \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3^k+1 & 3^k-1 \\ 3^k-1 & 3^k+1 \end{pmatrix}.$$

Problem 2: Consider the sequence of numbers f_0, f_1, f_2, ..., defined by the recurrence relation $f_{n+2} = 2f_{n+1} + 2f_n$, starting with $f_0 = f_1 = 1$ (giving 1, 1, 4, 10, 28, 76, 208, ...).

(a) As we did for the Fibonacci numbers in class (and in the book), express this process as repeated multiplication of a vector $\vec{u}_k = (f_{k+1}, f_k)^{\mathrm{T}}$ by a matrix A: $\vec{u}_{k+1} = A\vec{u}_k$, and thus $\vec{u}_k = A^k\vec{u}_0$. What is A?

(b) Find the eigenvalues of A, and thus explain that the ratio f_{k+1}/f_k tends towards __________ as $k \to \infty$. Check this by computing f_{k+1}/f_k for the first few terms in the sequence.

(c) Give an explicit formula for f_k (it can involve powers of numbers, but not powers of matrices) by expanding f_0 in the basis of the eigenvectors of A.

(d) If we apply the recurrence relation in *reverse*, we use the formula: $f_n = f_{n+2}/2 - f_{n+1}$ (just solving the previous recurrence formula for f_n). Show that you get the *same* reverse formula if you just compute A^{-1}.

(e) What does $|f_k/f_{k+1}|$ tend towards as $k \to -\infty$ (i.e. after we apply the formula in reverse many times)? (Very little calculation required!)

Solution (30 points = 5+5+10+5+5)

(a) The recurrence relation gives

$$\begin{aligned} f_{k+2} &= 2f_{k+1} + 2f_k \\ f_{k+1} &= f_{k+1}. \end{aligned}$$

Another way to write this is

$$\vec{u}_{k+1} = \begin{pmatrix} f_{k+2} \\ f_{k+1} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix} = A\vec{u}_k, \quad \Rightarrow \quad A = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}.$$

(b) Solving $\det(A - \lambda I) = \lambda^2 - 2\lambda - 2 = 0$ gives $\lambda_1 = 1 + \sqrt{3} \approx 2.732$ and $\lambda_2 = 1 - \sqrt{3} \approx -0.732$. Since $|\lambda_1| > |\lambda_2|$, f_{k+1}/f_k tends towards $\lambda_1 \approx 2.732$, as $k \to \infty$.

Check first few terms: $1, 1, 4, 10, 28, 76, 208, 568, \ldots$

$$\begin{aligned} 1/1 &= 1 \\ 4/1 &= 4 \\ 10/4 &= 2.5 \\ 28/10 &= 2.8 \\ 76/28 &\approx 2.7143 \\ 208/76 &\approx 2.7368 \\ 568/208 &\approx 2.7308. \end{aligned}$$

REMARK: Another phenomenon one should notice is that, in the sequence, $f_{k+1}/f_k > \lambda_1$ when k is odd and $f_{k+1}/f_k < \lambda_1$ when k is even. This is because the second eigenvalue λ_2 is negative. We will see in the explicit formula of f_k below.

(c) Find the eigenvectors.

$$\lambda_1 = 1+\sqrt{3}, \quad A - \lambda I = \begin{pmatrix} 1-\sqrt{3} & 2 \\ 1 & -1-\sqrt{3} \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1+\sqrt{3} \\ 1 \end{pmatrix}$$
$$\lambda_2 = 1-\sqrt{3}, \quad A - \lambda I = \begin{pmatrix} 1+\sqrt{3} & 2 \\ 1 & -1+\sqrt{3} \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1-\sqrt{3} \\ 1 \end{pmatrix}.$$

Then, we expand $\vec{u}_0$ as follows.

$$\vec{u}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\sqrt{3} \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1-\sqrt{3} \\ 1 \end{pmatrix} = \frac{1}{2}v_1 + \frac{1}{2}v_2.$$

Hence, we have

$$\vec{u}_k = A^k \vec{u}_0 = \frac{1}{2}\lambda_1^k v_1 + \frac{1}{2}\lambda_2^k v_2.$$

In particular,

$$f_k = \frac{1}{2}\left[(1+\sqrt{3})^k + (1-\sqrt{3})^k\right].$$

REMARK: From the explicit formula for f_k, we see that $f_k > \frac{1}{2}(1+\sqrt{3})^k$ if k is even and $f_k < \frac{1}{2}(1+\sqrt{3})^k$ if k is odd. This explains the earlier remark that $f_{k+1}/f_k < 1+\sqrt{3}$ if k is even and $f_{k+1}/f_k > 1+\sqrt{3}$ if k is odd.

(d) It is easy to compute $A^{-1} = -\frac{1}{2}\begin{pmatrix} 0 & -2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1/2 & -1 \end{pmatrix}$.

Applying the recurrence relation in reverse gives

$$\begin{aligned} f_{n+1} &= f_{n+1} \\ f_n &= f_{n+2}/2 - f_{n+1}. \end{aligned}$$

That is

$$\vec{u}_n = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1/2 & -1 \end{pmatrix}\begin{pmatrix} f_{n+2} \\ f_{n+1} \end{pmatrix} = A^{-1}\vec{u}_{n+1}.$$

(e) Applying the process in reverse is dominated by the biggest-magnitude eigenvalue of A^{-1}. The eigenvalues of A^{-1} are just the reciprocals of the eigenvalues of

A, so its biggest-magnitude eigenvalue is the reciprocal of the smallest-magnitude eigenvalue of A, i.e. $1/\lambda_2$. Hence,

$$\left|\frac{f_k}{f_{k+1}}\right| \approx \left|\frac{\lambda_2^k}{\lambda_2^{k+1}}\right| = |\lambda_2^{-1}| = \frac{\sqrt{3}+1}{2} \approx 1.366.$$

Problem 3: Suppose that $A = S\Lambda S^{-1}$. Take determinants to prove that $\det A$ is the product of the eigenvalues of A. (This quick proof only works when A is ____________________________.)

Solution (5 points)

Since the determinant is multiplicative, we have

$$\det(A) = \det(S\Lambda S^{-1}) = \det(S)\det(\Lambda)\det(S)^{-1} = \det(\Lambda) = \lambda_1 \cdots \lambda_n,$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A.

The proof only works when A is *diagonalizable*.

REMARK: Note that the determinant is always the product of the eigenvalues, even for non-diagonalizable matrices. However, the proof for the non-diagonalizable case is a bit trickier.

Problem 4: In this problem, you will show that the trace of a matrix (the sum of the diagonal entries) is equal to the sum of the eigenvalues, by first showing that AB and BA have the *same trace* for any matrices A and B. Follow the following steps:

(a) The explicit formula for the entries c_{ij} of $C = AB$ is $c_{ij} = \sum_k a_{ik}b_{kj}$ (where a_{ik} and b_{kj} are the entries of A and B, respectively). The trace of C is $\sum_i c_{ii}$. Write down the explicit formula for the entries d_{ij} of the product $D = BA$. By plugging these matrix-multiply formulas into the formulas for the trace of $C = AB$ and $D = BA$, and comparing them, prove that AB and BA have the same trace.

(b) $A = S\Lambda S^{-1}$, assuming A is ____________________________. Combining this factorization with the fact you proved in (a), show that the trace of A is the same as the trace of Λ, which is sum of the eigenvalues.

Solution (10 points)

(a) An explicit formula for entries d_{ij} of D is $d_{ij} = \sum_k b_{ik}a_{kj}$, just switching the role of a and b in the expression of c_{ij}. So,

$$\text{trace}(C) = \sum_i c_{ii} = \sum_i \sum_k a_{ik}b_{ki}$$
$$\text{trace}(D) = \sum_i d_{ii} = \sum_i \sum_k b_{ki}a_{ik}.$$

They are the same because we can change the indices i and k in the summation.

(b) For this to work, we have to assume that A is a *diagonalizable* $n \times n$ matrix. Using the identity above, we have (by viewing $S\Lambda$ as one matrix)

$$\text{trace}(A) = \text{trace}(S\Lambda S^{-1}) = \text{trace}(S^{-1}S\Lambda) = \text{trace}(\Lambda) = \lambda_1 + \cdots + \lambda_n,$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A.

REMARK: This again works even if A does not have a full set of independent eigenvectors. One may use generalized eigenvectors to do the same trick. We again will omit the details.

Problem 5: Suppose $A^2 = A$. (This does *not* mean $A = I$, since A might not be invertible; it might be a projection onto a subspace, for example.)

(a) Explain why any eigenvector with $\lambda = 0$ is in the ____________________ space of A, *and vice versa* (any nonzero vector in that space is an eigenvector with $\lambda = 0$).

(b) Explain why any eigenvector with $\lambda = 1$ is in the ____________________ space of A, *and vice versa* (any nonzero vector in that space is an eigenvector with $\lambda = 1$). (Hint: first explain why each column of A is an eigenvector.)

(c) Conclude from the dimensions of these subspaces that any such A must have a full set of independent eigenvectors and hence be diagonalizable.

Solution (15 points = 5+5+5)

(a) Any eigenvector with $\lambda = 0$ is in the *nullspace* of A. This is because $\vec{v} \in N(A)$ if and only if $A\vec{v} = 0 = 0 \cdot \vec{v}$.

(b) Any eigenvector with $\lambda = 1$ is in the *column space* of A. For each column $\vec{x}$ of A, $A^2 = A$ implies that $A\vec{x} = \vec{x}$. Hence, each column vector is an eigenvector with $\lambda = 1$, so is any vector in the column space.

Conversely, if v is an eigenvector with $\lambda = 1$, that means that $Av = v$, which implies that v is in $C(A)$.

REMARK: Note that for a more general matrix with nonzero eigenvalues $\neq 1$, it is still true that any eigenvector for a nonzero eigenvalue is in the column space of A, since $A(v/\lambda) = v$. However, the span of these eigenvectors may only be a subspace of the column space (if A is not diagonalizable).

(c) Say that A is an $n \times n$ matrix. Note that the dimension of the column space is the rank of A, whereas the dimension of the nullspace of A is $n - \text{rank}(A)$. The dimensions add up to n. Hence, the eigenspace for $\lambda = 0$ and the eigenspace for λ_0 span the whole space $\mathbb{R}^n$. Therefore, A much have a full set of independent eigenvectors and hence diagonalizable.

Problem 6: A genderless alien society survives by cloning/budding. Every year, 5% of young aliens become old, 3% of old aliens become dead, and 1% of the old aliens and 2% of the dead aliens are cloned into new young aliens. The population can be described by a Markov process:

$$\begin{pmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{pmatrix}_{\text{year } k+1} = A \begin{pmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{pmatrix}_{\text{year } k}$$

(a) Give the Markov matrix A, and compute (without Matlab) the steady-state young/old/dead population fractions.

(b) In Matlab, enter your matrix A and a random starting vector `x = rand(3,1); x = x / sum(x)` (normalized to sum to 1). Now, compute the population for the first 100 years, and plot it versus time, by the following Matlab code:

```
p = [];
for k = 0:99
   p = [ p, A^k * x ];
end
plot([0:99], p')
legend('young', 'old', 'dead')
xlabel('year'); ylabel('population fraction');
```

Check that the final population `p(:,end)` is close to your predicted steady state.

(c) In Matlab, compute A to a large power A^{1000} (in Matlab: `A^1000`). Explain why you get what you do, in light of your answer to (a).

Solution (15 points = 5+5+5)

(a) The Markov matrix A is given by

$$\begin{pmatrix} 0.95 & 0.01 & 0.02 \\ 0.05 & 0.96 & 0 \\ 0 & 0.03 & 0.98 \end{pmatrix}$$

Computing the steady-state is equivalent to find the eigenvector for $\lambda = 1$. We do a Gaussian elimination.

$$A - I = \begin{pmatrix} -0.05 & 0.01 & 0.02 \\ 0.05 & -0.04 & 0 \\ 0 & 0.03 & -0.02 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -0.05 & 0.01 & 0.02 \\ 0 & -0.03 & 0.02 \\ 0 & 0.03 & -0.02 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 5 & -1 & -2 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

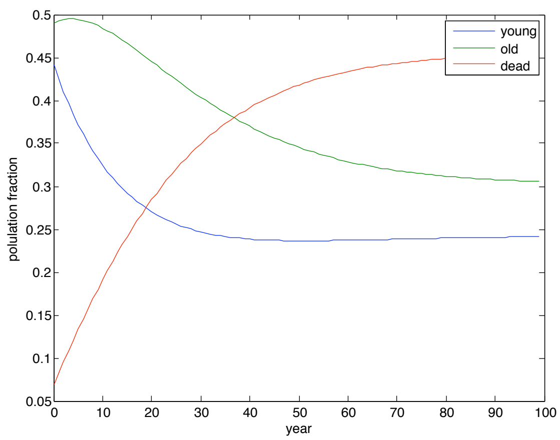
A solution is given by $v = (\frac{8}{5}, 2, 3)^{\mathrm{T}}$. If we normalize the vector so that the sum of the coordinates is 1, we get $v' = \frac{1}{33}(8, 10, 15)^{\mathrm{T}} \approx (0.2424, 0.3030, 0.4545)^{\mathrm{T}}$.

In other words, the steady-state young/old/dead population ratio is $8 : 10 : 15$.

(b) The code and the result is as follows.

```
>> A = [0.95, 0.01, 0.02; 0.05, 0.96, 0; 0, 0.03, 0.98]

A =

    0.9500    0.0100    0.0200
    0.0500    0.9600         0
         0    0.0300    0.9800

>> x = rand(3, 1); x = x / sum(x)

x =

    0.4410
    0.4903
    0.0687
```

```
>> p = [];
>> for k = 0:99
p = [p, A^k * x ];
end
>> plot([0:99], p')
>> legend('young', 'old', 'dead')
>> xlabel('year'); ylabel('polulation fraction');
>> p(:, end)

ans =

    0.2412
    0.3058
    0.4530
```



(c)

```
>> A^1000

ans =

    0.2424    0.2424    0.2424
    0.3030    0.3030    0.3030
    0.4545    0.4545    0.4545
```

For a large power like A^{1000}, we should expect any initial vector to converge to the steady state. That means that the column space of A^{1000} should just be the steady-state eigenvector, which means that each column of A should be approximately this eigenvector (normalized so that each column sums to 1).

Problem 7: If A is *both* a symmetric matrix and a Markov matrix, why is its steady-state eigenvector $(1, 1, \ldots, 1)^{\mathrm{T}}$?

Solution (5 points)

A very important property of a Markov matrix is that $[1\ 1\ \cdots\ 1\ 1]A = [1\ 1\ \cdots\ 1\ 1]$. Taking transpose, we have

$$A^{\mathrm{T}}[1\ 1\ \cdots\ 1\ 1]^{\mathrm{T}} = [1\ 1\ 1\ \cdots\ 1\ 1]^{\mathrm{T}}.$$

But $A = A^{\mathrm{T}}$ is symmetric. Hence, $[1\ 1\ \ldots,\ 1]^{\mathrm{T}}$ is an eigenvector with $\lambda = 1$. It is a steady-state eigenvector.

Problem 8: Find the λ's and $\vec{x}$'s so that $\vec{u} = e^{\lambda t}\vec{x}$ is a solution of

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} \vec{u}. \tag{1}$$

Make a linear combination of these solutions to solve this equation with the initial condition $\vec{u}(0) = (5, -2)^{\mathrm{T}}$.

Solution (15 points)

Step 1: find the eigenvalues and the eigenvectors of the matrix $A = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}$. Solving $\det(A - \lambda I) = \lambda^2 - \lambda - 2 = 0$ gives $\lambda_1 = 2$ and $\lambda_2 = -1$.

$$\lambda_1 = 2, \quad A - \lambda I = \begin{pmatrix} 0 & 3 \\ 0 & -3 \end{pmatrix}, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = -1, \quad A - \lambda I = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Step 2: we have

$$\vec{u}_1 = e^{\lambda_1 t}\vec{v}_1 = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{u}_2 = e^{\lambda_2 t}\vec{v}_2 = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$$

to be the solution of (1).

Step 3: solve the initial value problem for $\vec{u} = c_1\vec{u}_1 + c_2\vec{u}_2$; this requires to solve

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}.$$

We then have $c_1 = 3, c_2 = 2$. Hence,

$$\vec{u} = c_1\vec{u}_1 + c_2\vec{u}_2 = \begin{pmatrix} 3e^{2t} + 2e^{-t} \\ -2e^{-t} \end{pmatrix}.$$

Problem 9: Explain how to write an equation $\alpha\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + \gamma y = 0$ as a vector equation $M\frac{d\vec{u}}{dt} = A\vec{u}$.

Solution (5 points)

Write $\vec{u} = \begin{pmatrix} \frac{dy}{dt} \\ y \end{pmatrix}$. Then $\frac{d\vec{u}}{dt} = \begin{pmatrix} \frac{d^2y}{dt^2} \\ \frac{dy}{dt} \end{pmatrix}$. The equation gives that

$$\begin{aligned} \alpha\frac{d^2y}{dt^2} &= -\beta\frac{dy}{dt} - \gamma y, \\ \frac{dy}{dt} &= \frac{dy}{dt}. \end{aligned}$$

This translates to say

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \frac{d\vec{u}}{dt} = \begin{pmatrix} -\beta & -\gamma \\ 1 & 0 \end{pmatrix}.$$

Remark to the graders, it is okay for the students to assume that $\alpha \neq 0$ and write the final result as follows.

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} -\frac{\beta}{\alpha} & -\frac{\gamma}{\alpha} \\ 1 & 0 \end{pmatrix}.$$

Problem 10: A matrix A is antisymmetric, or "skew" symmetric, which means that $A^{\mathrm{T}} = -A$. Prove that the matrix $Q = e^{At}$ is orthogonal: transpose the series for $Q = e^{At}$ to show that you get the series for e^{-At}, and thus $Q^{\mathrm{T}}Q = I$. Therefore, if $\vec{u}(t) = e^{At}\vec{u}(0)$ is *any* solution to the system $\frac{d\vec{u}}{dt} = A\vec{u}$, then we know that $\|\vec{u}(t)\|/\|\vec{u}(0)\| =$ __________.

Solution (15 points=10+5)

Write out the series that defines $Q = e^{At}$ and transpose.

$$Q^{\mathrm{T}} = (e^{At})^{\mathrm{T}} = \Big(\sum_{n=0}^{\infty} \frac{(At)^n}{n!}\Big)^{\mathrm{T}} = \sum_{n=0}^{\infty} \frac{(A^{\mathrm{T}}t)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-At)^n}{n!} = e^{-At}.$$

Hence $Q^{\mathrm{T}}Q = e^{-At}e^{At} = I$. This implies that Q is orthogonal.

Since orthogonal matrices preserve norms, we must have $\|e^{At}\vec{u}(0)\| = \|u(0)\|$. Hence, $\|\vec{u}(t)\|/\|\vec{u}(0)\| = 1$.

REMARK: it is not in general true that $e^{Bt}e^{At} = e^{(B+A)t}$ for any square matrices A and B. In fact, this is only true if $AB = BA$. But this is certainly true for $B = -A$ as it is here. More simply, e^{At} is the matrix that propagates the solution forward in time by t, while e^{-At} propagates the solution backwards in time by $-t$, so the two matrices must be inverses.

Problem 11: If $A^2 = A$, show from the infinite series that $e^{At} = I + (e^t - 1)A$. For $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, this gives $e^{At} =$ __________.

Solution (10 points)

Since $A^2 = A$, we have $A^k = A$ for any $k > 0$.

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = I + \sum_{n=1}^{\infty} \frac{At^n}{n!} = I + A(\sum_{n=1}^{\infty} \frac{t^n}{n!}) = I + A(e^t - 1).$$

When $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, we have

$$e^{At} = I + (e^t - 1)A = \begin{pmatrix} e^t & e^t - 1 \\ 0 & 1 \end{pmatrix}$$

Problem 12: Assume A is diagonalizable with real eigenvalues. What condition on the eigenvalues of A ensures that the solutions of $\frac{d\vec{u}}{dt} = A\vec{u}$ will *not* blow up for $t \to \infty$? In comparison, what condition on the eigenvalues of A ensures that solutions of the linear recurrence relation $\vec{u}_{k+1} = A\vec{u}_k$ will *not* blow up for $k \to \infty$?

Solution (10 points = 5+5)

For ODE problem, if the solutions do not blow up as $t \to \infty$, the eigenvalues λ of A has to have real part less than or equal to 0, i.e. $\mathrm{Re}(\lambda) \leq 0$. This is because the solution looks like $e^{\lambda t}\vec{v}$, where $\vec{v}$ is an eigenvector with eigenvalue λ.

For the linear recurrence problem, if $\vec{u}_k$ does not blow up, the eigenvalues λ of A has to have absolute value less than or equal to 1, that is $|\lambda_1| \leq 1$. This is because the main term in $\vec{u}_k$ looks like $\lambda^k\vec{v}$, where $\vec{v}$ is an eigenvector with eigenvalue λ.