18.06 Problem Set 6

Due Wednesday, 8 April 2009 at 4pm in 2-106.

1. If A is a 7×7 matrix and $\det A = 17$, what is $\det(3A^2)$?

2. The determinant of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det A = ad - bc$. Assuming no row swaps are required, perform elimination on A and show explicitly that $ad - bc$ is the product of the pivots.

3. If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$ ($n > 1$), what is the determinant of $\mathbf{x}\mathbf{y}^T$? (This is *not* the dot product $\mathbf{x}^T\mathbf{y}$.) Hint: the rank of $\mathbf{x}\mathbf{y}^T$ is ______.

4. Does $\det(AB) = \det(BA)$ in general? **(a)** True or false if A and B are square $n\times n$ matrices? **(b)** True or false if A is $m\times n$ and B is $n\times m$, with $m \neq n$? For both (a) and (b), give a reason if true or a counter-example if false.

5. Find the eigenvalues of the matrices $A = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}$, $A^2 = \begin{pmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{pmatrix}$, $A^\infty \approx \begin{pmatrix} 0.57143 & 0.57143 \\ 0.42857 & 0.42857 \end{pmatrix}$, and $B = \begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix}$. Note that B is just A with the rows exchanged, which may change λ!

6. A singular square matrix must have an eigenvalue of $\lambda =$ ______.

7. The matrix $A = \begin{pmatrix} 2 & 10 & -2 \\ 10 & 5 & 8 \\ -2 & 8 & 11 \end{pmatrix}$ has the three eigenvalues $\lambda = 18, 9, -9$.

   (a) Find eigenvectors corresponding to these three eigenvalues.

   (b) Compute the dot products of the eigenvectors you found with one another. Hence, the eigenvectors divided by their lengths form an ____________________ basis with this A!

   (c) Write the vector $\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$ in the basis of your three eigenvectors, and thereby compute $A^{10}\mathbf{x}$ (write your answer as a summation of eigenvectors times λ^{10} for each λ).

8. The eigenvalues of A and A^T are the same, because $\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I)$. By coming up with a 2×2 counter-example, show that the eigen*vectors* of A and A^T need *not* be the same.

9. In Matlab, make a random 5×5 symmetric matrix A by the commands:

   ```
   A = rand(5,5); A = A' * A;
   B = A
   ```

   copying the result to a matrix B. Now, you will repeatedly compute the QR factorization $B = QR$ and then replace B with the new matrix RQ, via the commands:

   ```
   [Q,R] = qr(B); B = R * Q
   ```

   Repeat the above line over and over (you can use the up-arrow key in Matlab to fetch the previous command), until B stops changing. You can ignore tiny numbers smaller than 10^{-7} (which Matlab prints as `1e−7`) or so.

   You should find that B converges towards a diagonal matrix! Compare the numbers on its diagonal [`diag(B)` in Matlab] to the eigenvalues of A [computed by `eig(A)` in Matlab].

10. If we perform the QR factorization of a square matrix A, obtaining $A = QR$, show that the matrix RQ is *similar* to A (as defined in section 6.6) and hence has the same eigenvalues (hint: $R = Q^T A$, and Q is an ____________________ matrix). Thus, the eigenvalues of the matrix B in the previous problem are the same as the eigenvalues of A, no matter how many times you do the $QR \to RQ$ replacement.