18.06 Problem Set 6 Solution

Due Wednesday, 8 April 2009 at 4 pm in 2-106.

Total: 100 points.

Problem 1: If A is a 7×7 matrix and $\det A = 17$, what is $\det(3A^2)$?

$\boxed{\text{Solution}}$ (10 points)

For any square matrix M, N of the same size, we have $\det(MN) = \det(M)\det(N)$. Thus, $\det(A^2) = \det(A)\det(A) = 17^2$.

Note that we proved in class that multiplying a single row by 3 multiplies the determinant by 3. Multiplying the whole 7×7 matrix by 3 multiplies all 7 rows by 3, and hence multiplies the determinant by 3^7. Hence, we have $\det(3A^2) = 3^7 \cdot 17^2 = 632043$. (It is okay to leave it as $3^7 \cdot 17^2$.

Problem 2: The determinant of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det A = ad - bc$. Assuming no row swaps are required, perform elimination on A and show explicitly that $ad - bc$ is the product of the pivots.

$\boxed{\text{Solution}}$ (5 points)

We need to subtract c/a times row 1 from row 2, that leaves us

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} a & b \\ 0 & d - b \cdot a/c \end{pmatrix}$$

The product of the pivots is $a \cdot (d - ab/c) = ad - bc = \det A$.

REMARK: We proved that this is true for any $n \times n$ matrix in class (it is also in the book), not including row swaps (which flip the sign).

Problem 3: If $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^n$ $(n > 1)$, what is the determinant of $\vec{x}\vec{y}^{\mathrm{T}}$? (This is *not* the dot product $\vec{x}^{\mathrm{T}}\vec{y}$.) Hint: the rank of $\vec{x}\vec{y}^{\mathrm{T}}$ is ______.

$\boxed{\text{Solution}}$ (5 points) The product $A = \vec{x}\vec{y}^{\mathrm{T}}$ is an $n \times n$ matrix. But we know its column space $C(A) \subseteq C(\vec{x})$ is at most 1-dimensional (spanned by $\vec{x}$ if $\vec{y} \neq \vec{0}$). Hence A is singular because $n > 1 \geq \mathrm{rank}(A) = \dim C(A)$. Therefore, $\det(A) = 0$.

An alternative way to see it is that all the rows are a multiple of $\vec{y}^{\mathrm{T}}$, and all the columns are multiples of $\vec{x}$. Hence, the determinant must be zero because the determinant is zero when any rows or columns are linearly dependent.

Problem 4: Does $\det(AB) = \det(BA)$ in general? **(a)** True or false if A and B are square $n \times n$ matrices? **(b)** True or false if A is $m \times n$ and B is $n \times m$, with $m \neq n$? For both (a) and (b), give a reason if true or a counter-example if false.

$\boxed{\text{Solution}}$ (10 points = 5+5)

(a) True. This is because if A and B are square matrix, $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$.

(b) False. This is almost always not true. We may use the solution to problem 3 to make a counter example: $\det(xy^{\mathrm{T}}) = 0$, but $\det(x^{\mathrm{T}}y) = x^{\mathrm{T}}y$. For example,

$$A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Then,

$$AB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad BA = (2);$$
$$\det(AB) = 0, \quad \det(BA) = 2.$$

REMARK: In fact, if $m > n$, $\det(AB) = 0$. This is because the column space $C(AB) \subseteq C(A)$ has dimension $\leq n < m$. So, AB, as an $m \times m$ matrix, is singular. Hence $\det(AB) = 0$.

Problem 5: Find the eigenvalues of the matrices $A = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}$, $A^2 = \begin{pmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{pmatrix}$, $A^\infty \approx \begin{pmatrix} 0.57143 & 0.57143 \\ 0.42857 & 0.42857 \end{pmatrix}$, and $B = \begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix}$. Note that B is just A with the rows exchanged, which may change λ!

$\boxed{\text{Solution}}$ (15 points = 5+2.5+2.5+5)

(a) For the matrix A: $\det(A - \lambda I) = \lambda^2 - 1.3\lambda + 0.3 = 0$. We solve to have $\lambda_1 = 1$, $\lambda_2 = 0.3$.

(b) For the matrix A^2: $\det(A^2 - \lambda I) = \lambda^2 - 1.09\lambda + 0.09 = 0$. We solve to have $\lambda_1 = 1$, $\lambda_2 = 0.09$.

(c) For the matrix A^∞: $\det(A^\infty - \lambda I) = \lambda^2 - \lambda = 0$. We solve to have $\lambda_1 = 1$, $\lambda_2 = 0$.

(d) For the matrix B: $\det(B - \lambda I) = \lambda^2 - 0.7\lambda - 0.3 = 0$. We solve to have $\lambda_1 = 1$, $\lambda_2 = -0.3$.

REMARK: It is discussed in class that all powers of A have the same eigenvectors, and the eigenvalues are simply exponentiated. In particular, the eigenvalues in (b) are the squares of the eigenvalues in (a), and the eigenvalues in (c) are just the limit of an infinite power of the eigenvalues in (a).

Problem 6: A singular square matrix must have an eigenvalue of $\lambda =$ ______.

$\boxed{\text{Solution}}$ (5 points)

A singular square matrix must have an eigenvalue of $\lambda = 0$. This is because pick any vector $\vec{x}$ in the nullspace of the matrix A. Then $A\vec{x} = 0 = 0 \cdot \vec{x}$ implies that $\vec{x}$ is an eigenvector of A with eigenvalue 0.

In general, a matrix A being singular is equivalent to saying that $N(A)$ contains nonzero vectors (which is equivalent to saying A has dependent columns).

Problem 7: The matrix $A = \begin{pmatrix} 2 & 10 & -2 \\ 10 & 5 & 8 \\ -2 & 8 & 11 \end{pmatrix}$ has the three eigenvalues $\lambda = 18, 9, -9$.

(a) Find eigenvectors corresponding to these three eigenvalues.

(b) Compute the dot products of the eigenvectors you found with one another. Hence, the eigenvectors divided by their lengths form an __________________ basis with this A!

(c) Write the vector $\vec{x} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^{\mathrm{T}}$ in the basis of your three eigenvectors, and thereby compute $A^{10}\vec{x}$ (write your answer as a summation of eigenvectors times λ^{10} for each λ).

$\boxed{\text{Solution}}$ (25 points = 10+5+10)

The general principal is that we find the eigenvectors by finding the nullspace of $A - \lambda I$, and we find the nullspace by doing elimination to row-reduced echelon form as we learned before exam 1. Once we find the nullspace, any special solution is an eigenvector; usually there is only one special solution of $A - \lambda I$ (the nullspace is 1-dimensional). Note that any multiple of an eigenvector is also an eigenvector with the same eigenvalue, but it is only necessary to find one.

(a) For $\lambda_1 = 18$, we do elimination to echelon form to find the nullspace of the matrix

$$A - \lambda_1 I = \begin{pmatrix} -16 & 10 & -2 \\ 10 & -13 & 8 \\ -2 & 8 & -7 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -2 & 8 & -7 \\ 10 & -13 & 8 \\ -16 & 10 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -2 & 8 & -7 \\ 0 & 27 & -27 \\ 0 & -54 & 54 \end{pmatrix}$$
$$\rightsquigarrow \begin{pmatrix} -2 & 8 & -7 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

So the first eigenvector is a vector in the nullspace of the above matrix

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

We can replace this by any multiples of v_1.

For $\lambda_2 = 9$,

$$A - \lambda_2 I = \begin{pmatrix} -7 & 10 & -2 \\ 10 & -4 & 8 \\ -2 & 8 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -2 & 8 & 2 \\ 10 & -4 & 8 \\ -7 & 10 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -2 & 8 & 2 \\ 0 & 36 & 18 \\ 0 & -18 & -9 \end{pmatrix}$$
$$\rightsquigarrow \begin{pmatrix} -1 & 4 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

So the second eigenvector is

$$v_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

For $\lambda_3 = -9$,

$$A - \lambda_3 I = \begin{pmatrix} 11 & 10 & -2 \\ 10 & 14 & 8 \\ -2 & 8 & 20 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -2 & 8 & 20 \\ 10 & 14 & 8 \\ 11 & 10 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -2 & 8 & 20 \\ 0 & 54 & 108 \\ 0 & 54 & 108 \end{pmatrix}$$
$$\rightsquigarrow \begin{pmatrix} -1 & 4 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

So the third eigenvector is

$$v_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

(b) Orthonormal basis.

$$\begin{aligned}
v_1^{\mathrm{T}} v_2 &= 1 \cdot 2 + 2 \cdot 1 + 2 \cdot (-2) = 0, \\
v_1^{\mathrm{T}} v_3 &= 1 \cdot 2 + 2 \cdot (-2) + 2 \cdot 1 = 0, \\
v_2^{\mathrm{T}} v_3 &= 2 \cdot 2 + 1 \cdot (-2) + (-2) \cdot 1 = 0.
\end{aligned}$$

So, the eigenvectors are orthogonal to each other. Hence, the eigenvectors divided by their lengths form an *orthonormal* basis with A.

REMARK: In general, the eigenvectors of any symmetric matrix A with distinct eigenvalues are always orthogonal; we will prove this in class for lecture 27.

(c) Since v_1, v_2, v_3 are orthogonal, we can find an orthonormal basis out of it by taking

$$q_1 = v_1/\|v_1\| = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \quad q_2 = v_2/\|v_2\| = \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}, \quad q_3 = v_3/\|v_3\| = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}.$$

Let Q be the matrix whose columns are these orthonormal basis, i.e.

$$Q = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}.$$

Write $\vec{x} = Qc$. Then, we have

$$c = Q^{\mathrm{T}} \vec{x} = Q = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

Hence,

$$\begin{aligned}
A^{10}\vec{x} &= c_1 A^{10} q_1 + c_2 A^{10} q_2 + c_3 A^{10} q_3 = c_1 \lambda_1^{10} q_1 + c_2 \lambda^2 q_2 + c_3 \lambda_3^{10} q_3 \\
&= \frac{1}{3} \cdot 18^{10} \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix} + \frac{2}{3} \cdot 9^{10} \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix} + \frac{2}{3} \cdot (-9)^{10} \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}.
\end{aligned}$$

Problem 8: The eigenvalues of A and A^{T} are the same, because $\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I)$. By coming up with a 2×2 counter-example, show that the eigen*vectors* of A and A^T need *not* be the same.

$\boxed{\text{Solution}}$ (10 points)

For example, $A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$. We solve $\det(A - \lambda I) = \lambda^2 - 4 = 0$. So the eigenvalues are $\lambda_1 = 2, \lambda_2 = -2$.

For $\lambda_1 = 2$,

$$A - \lambda_1 I = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \quad \Rightarrow \quad v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$A^{\mathrm{T}} - \lambda_1 I = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \quad \Rightarrow \quad v_1' = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

They are not the same.

REMARK: A and A^{T} here are actually similar: $A^{\mathrm{T}} = P^{-1}AP$, where P is the permutation matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is why the eigenvectors of A^{T} are P^{-1} $(= P)$ times the eigenvectors of A, i.e. the rows of the eigenvectors are simply swapped. More generally, it is possible to prove that every square matrix is similar to its transpose (and the proof is pretty easy if you assume A is diagonalizable), so one can write an explicit formula relating the eigenvectors of A and A^{T}. We'll leave this as an exercise for the interested student.

Problem 9: In Matlab, make a random 5×5 symmetric matrix A by the commands:

```
A = rand(5,5); A = A' * A;
B = A
```

copying the result to a matrix B. Now, you will repeatedly compute the QR factorization $B = QR$ and then replace B with the new matrix RQ, via the commands:

```
[Q,R] = qr(B); B = R * Q
```

Repeat the above line over and over (you can use the up-arrow key in Matlab to fetch the previous command), until B stops changing. You can ignore tiny numbers smaller than 10^{-7} (which Matlab prints as `1e−7`) or so.

You should find that B converges towards a diagonal matrix! Compare the numbers on its diagonal [`diag(B)` in Matlab] to the eigenvalues of A [computed by `eig(A)` in Matlab].

$\boxed{\text{Solution}}$ (10 points)

The code using QR $\to$ RQ replacement to compute the eigenvalue is as follows.

```
>> A = rand(5, 5); A = A' * A;
>> B = A

B =

    2.7345    1.8859    2.0785    1.9442    1.9567
    1.8859    2.2340    2.0461    2.3164    2.0875
    2.0785    2.0461    2.7591    2.4606    1.9473
    1.9442    2.3164    2.4606    2.5848    2.2768
    1.9567    2.0875    1.9473    2.2768    2.4853

>> [Q,R] = qr(B); B = R * Q

B =

   10.7030    1.5894    0.3277    0.1970   -0.0008
    1.5894    1.1296    0.1032    0.1190   -0.0004
    0.3277    0.1032    0.6661   -0.1293    0.0007
    0.1970    0.1190   -0.1293    0.2986   -0.0016
   -0.0008   -0.0004    0.0007   -0.0016    0.0004

>> [Q,R] = qr(B); B = R * Q

B =

   10.9735    0.1349    0.0166    0.0045    0.0000
    0.1349    0.8890    0.0183    0.0244    0.0000
    0.0166    0.0183    0.6936   -0.0534   -0.0000
    0.0045    0.0244   -0.0534    0.2413    0.0000
    0.0000    0.0000   -0.0000    0.0000    0.0004

>> [Q,R] = qr(B); B = R * Q
```

```
B =

   10.9753    0.0110    0.0010    0.0001   -0.0000
    0.0110    0.8886    0.0122    0.0064   -0.0000
    0.0010    0.0122    0.6986   -0.0184    0.0000
    0.0001    0.0064   -0.0184    0.2348   -0.0000
   -0.0000   -0.0000    0.0000   -0.0000    0.0004

>> [Q,R] = qr(B); B = R * Q

B =

   10.9753    0.0009    0.0001    0.0000   -0.0000
    0.0009    0.8890    0.0094    0.0017    0.0000
    0.0001    0.0094    0.6990   -0.0062   -0.0000
    0.0000    0.0017   -0.0062    0.2341    0.0000
    0.0000    0.0000   -0.0000    0.0000    0.0004

....... (repeated 21 times)

>> [Q,R] = qr(B); B = R * Q

B =

   10.9753    0.0000    0.0000    0.0000   -0.0000
    0.0000    0.8894    0.0000   -0.0000   -0.0000
    0.0000    0.0000    0.6986   -0.0000   -0.0000
    0.0000    0.0000   -0.0000    0.2340    0.0000
    0.0000    0.0000   -0.0000    0.0000    0.0004
```

The eigenvalue of A computed by `eig(A)` is

```
>> eig(A)

ans =

    0.0004
    0.2340
    0.6986
    0.8894
```

```
10.9753
```

They give the same answer. Also note that diagonal of B is the eigenvalues in descending order. (If the eigenvalues were positive and negative, it would be in descending order by $|\lambda|$.

REMARK: This is the basic step in what is known as the "QR" algorithm to compute the eigenvalues of a matrix (and it can also get eigenvectors at the same time with some modifications), which was first discovered in 1959 and was named one of the "top 10 algorithms of the century" in 2000. Matlab's "eig" function contains a more sophisticated version of the same algorithm, modified in various ways to make it converge more quickly. The proof of *why* it converges is rather subtle; it turns out that you are *implicitly* doing a Gram-Schmidt orthogonalization of A^k after k steps of the QR algorithm. Further discussion of the QR algorithm is a topic for another course (18.335), however.

Problem 10: If we perform the QR factorization of a square matrix A, obtaining $A = QR$, show that the matrix RQ is *similar* to A (as defined in section 6.6) and hence has the same eigenvalues (hint: $R = Q^T A$, and Q is an __________________________ matrix). Thus, the eigenvalues of the matrix B in the previous problem are the same as the eigenvalues of A, no matter how many times you do the $QR \rightarrow RQ$ replacement.

$\boxed{\text{Solution}}$ (5 points)

The matrix Q is an orthogonal matrix. Hence $Q^{-1} = Q^{\mathrm{T}}$.

The matrices RQ and A are therefore similar because $R = Q^{\mathrm{T}}A = Q^{-1}A$ implies $RQ = Q^{-1}AQ$.

REMARK: It is worthwhile to point out that RQ is also *symmetric*! This is because $RQ = Q^{\mathrm{T}}AQ$. More precisely, $(Q^{\mathrm{T}}AQ)^{\mathrm{T}} = Q^{\mathrm{T}}A^{\mathrm{T}}(Q^{\mathrm{T}})^{\mathrm{T}} = Q^{\mathrm{T}}AQ$.