18.06 Problem Set 3 Solution

Due Wednesday, 25 February 2009 at 4 pm in 2-106.
Total: 160 points.

Problem 1: Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 4 & 1 \\ 2 & 6 & 3 & 11 & 1 \\ 1 & 4 & 2 & 7 & 0 \end{pmatrix}$$

(a) Reduce A to echelon form U, find a special solution for each free variable, and hence describe all solutions to $Ax = 0$.

(b) By further row operations on U, find the reduced echelon form R.

(c) True or false: $N(R) = N(U)$?

(d) True or false: $C(A) = C(U)$?

Solution (25 points = 10+5+5+5)
(a) Use Gaussian elimination.

$$A = \begin{pmatrix} 1 & 2 & 1 & 4 & 1 \\ 2 & 6 & 3 & 11 & 1 \\ 1 & 4 & 2 & 7 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 & 4 & 1 \\ 0 & 2 & 1 & 3 & -1 \\ 0 & 2 & 1 & 3 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 & 4 & 1 \\ 0 & 2 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = U$$

Free variables are x_3, x_4, x_5. When $x_3 = 1, x_4 = 0, x_5 = 0$ we get a special solution

$$\begin{pmatrix} 0 \\ -1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

When $x_3 = 0, x_4 = 1, x_5 = 0$ we get a special solution

$$\begin{pmatrix} -1 \\ -3/2 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

When $x_3 = 0, x_4 = 0, x_5 = 1$ we get a special solution

$$\begin{pmatrix} -2 \\ 1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence the solution to $Ax = 0$ is

$$x = x_3 \begin{pmatrix} 0 \\ -1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -3/2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2 \\ 1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{for } x_3, x_4, x_5 \in \mathbb{R}.$$

(b) Continue using row operations, we have

$$U = \begin{pmatrix} 1 & 2 & 1 & 4 & 1 \\ 0 & 2 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1/2 & 3/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(c) Since U is obtained from R by row operations, they have the same null-space.

(d) No. For example, $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ lies in $C(A)$, but any elements in $C(U)$ has its third coordinates zero.

REMARK: In general, null-space $N(A)$ is invariant under invertible row operations. In contrast, column vector space $C(A)$ is invariant under invertible column operations. (Non-invertible operations in general may not preserve the spaces.)

Problem 2: If you do column elimination steps (instead of row eliminations) on a matrix A to get some other matrix U (like in problem 6 of pset 1), does $N(A) = N(U)$? Come up with a counter-example if false, or give an explanation why this should always hold if true.

Solution (10 points)

No. We can give a counter-example as follows.

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}.$$

Then, the null-space $N(A)$ of A is spanned by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$; in contrast, the null-space $N(U)$ of U is spanned by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. They are very different.

For invariance under row or column operations, please see the remark in previous problem.

Problem 3: Suppose that column 3 of a 4×6 matrix is all zero. Then x_3 must be a ____________ variable. Give one special solution for this matrix.

Solution (5 points)

The variable x_3 is a free variable. A special solution for this variable can be taken to be

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Problem 4: Fill in the missing numbers to make the matrix A rank 1, rank 2, and rank 3. (i.e. your solution should be three matrices).

$$A = \begin{pmatrix} & -3 & \\ 1 & 3 & -1 \\ & 9 & -3 \end{pmatrix}.$$

Solution (15 points = 5+5+5)

If we want A to have rank 1, we need to make the first and the third rows to be multiples of the second. This forces A to be

$$A_1 = \begin{pmatrix} -1 & -3 & 1 \\ 1 & 3 & -1 \\ 3 & 9 & -3 \end{pmatrix}$$

If we want A to have rank 2, we can, for example, make the first rows to be a multiple of the second, but not the third. For example, we may take

$$A_2 = \begin{pmatrix} -1 & -3 & 1 \\ 1 & 3 & -1 \\ 2 & 9 & -3 \end{pmatrix}$$

In other words, we change the lower-left entry of A_1 from 3 to 2.

For a randomly chosen A, it is very likely to be of rank 3 (full rank). We randomly use some 0's or 1's as the missing numbers, for example,

$$A_3 = \begin{pmatrix} 0 & -3 & 0 \\ 1 & 3 & -1 \\ 1 & 9 & -3 \end{pmatrix}.$$

Use Gaussian elimination, we have

$$\begin{pmatrix} 0 & -3 & 0 \\ 1 & 3 & -1 \\ 1 & 9 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & -1 \\ 0 & -3 & 0 \\ 1 & 9 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & -1 \\ 0 & -3 & 0 \\ 0 & 6 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Hence, it has rank 3.

Problem 5: Suppose A and B have the *same* reduced echelon form R. Therefore A equals a/an ________________ matrix multiplying B on the __________ (left or right).

Solution (5 points)

The matrix A equals to an invertible matrix multiplying B on the left. This is because the process of reducing to echelon form can be thought as multiplying row operation matrices on the left. So if two matrices A, B have the same echelon form, they can be written as $A = MR$ and $B = NR$, with M, N invertible. Hence $R = N^{-1}B$ and $A = MN^{-1}B$.

Problem 6: Write the complete solution (i.e. a particular solution plus all nullspace vectors) to the system:

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

Solution (10 points)

First step is to find the echelon form using Gaussian elimination

$$\begin{pmatrix} 1 & 3 & 1 & 2 & & 1 \\ 2 & 6 & 4 & 8 & & 3 \\ 0 & 0 & 2 & 4 & & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 1 & 2 & & 1 \\ 0 & 0 & 2 & 4 & & 1 \\ 0 & 0 & 2 & 4 & & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 1 & 2 & & 1 \\ 0 & 0 & 2 & 4 & & 1 \\ 0 & 0 & 0 & 0 & & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 0 & 0 & & 1/2 \\ 0 & 0 & 1 & 2 & & 1/2 \\ 0 & 0 & 0 & 0 & & 0 \end{pmatrix}$$

Free variables are x_2 and x_4. A particular solution for this system is $\begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix}$. The null-space is spanned by $\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$. So the solution to the system is

$$x = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} + x_2 \cdot \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \text{ for } x_2, x_4 \in \mathbb{R}.$$

Problem 7: Explain why these statements are *all false* by giving a counter-example for each:

(a) A system $Ax = b$ has at most one particular solution.

(b) A system $Ax = b$ has at least one particular solution.

(c) If there is only one special solution x_n in the nullspace and there exists some particular solution x_p, then the complete solution to $Ax = b$ is any linear combination of x_p and x_n.

(d) If A is invertible then there is no solution x_n the nullspace.

(e) The solution x_p with all free variables set to zero is the "shortest" solution (minimizing $\|x\|$).

Solution (25 points = 5+5+5+5+5)

(a) This is wrong because if we add any solution x_n in the null-space with any particular solution x_p, we will get a particular solution $x_p + x_n$ to the system. For example,

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, b = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

We may take x_p to be $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, or $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, or more generally, $\begin{pmatrix} 3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ for any $x_2 \in \mathbb{R}$.

(b) There could be no solution to the system at all. For example,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(c) This is false because "any linear combination" would include x_p multiplied by any constant, but only the nullspace vector x_n can be multiplied by any constant. For example, consider the matrix A from part (b) with a right-hand side $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In this case, a particular solution is $x_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the nullspace is spanned by the special solution $x_n = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The general solution is $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, where c is any constant. If we took all linear combinations of x_p and x_n, however, that would be $d \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ for any constants c and d, which is obviously not always a solution (for example, consider $c = d = 0$).

(d) There is always one solution $x_n = 0$ in the null-space.

(e) When the free variables are set to be zero has nothing to do with the length of $\|x\|$. For example,

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, b = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

We have

$$x = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

When setting x_2 to zero, we have $x_p = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ with $\|x_p\| = 3$; when setting $x_2 = 1$, we have $x'_p = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ with $\|x'_p\| = \sqrt{5} < 3$.

Problem 8: If A is a 3×7 matrix, its largest possible rank is __________. In this case, there is a pivot in every __________ of U and R, the solution to $Ax = b$ __________ (*always exists* or *is unique*), and the column space of A is __________. Construct an example of such a matrix A.

Solution (10 points)

3; row; always exists; $\mathbb{R}^3$.

Since the rank of A is smaller than the number of rows and the number of columns, rank$A \leq 3$. In this case, when we reduce it using Gaussian elimination, we will have 3 pivots and hence there is one on each row. The solution to $Ax = b$ would always exist and the column space is exactly $\mathbb{R}^3$. For example,

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 5 & 6 & 7 & 8 \\ 0 & 0 & 1 & 9 & 10 & 11 & 12 \end{pmatrix}.$$

REMARK: This is the full row rank case discussed in class.

Problem 9: If A is a 6×3 matrix, its largest possible rank is __________. In this case, there is a pivot in every __________ of U and R, the solution to $Ax = b$ __________ (*always exists* or *is unique*), and the nullspace of A is __________. Construct an example of such a matrix A.

Solution (10 points)

3; column; is unique (if exists); $\{0\}$.

Since the rank of A is smaller than the number of rows and the number of columns, rank$A \leq 3$. In this case, when we reduce it using column Gaussian elimination, we will have 3 pivots and hence there is one on each column. The solution to $Ax = b$ would be unique if it exists and the null space is $\{0\}$. For example,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

REMARK: This is the full column rank case discussed in class.

Problem 10: Find the rank of A, $A^{\mathrm{T}}A$, and AA^{T}, for $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}$.

Solution (15 points = 5+5+5)

Use Gaussian elimination to determine the rank.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \mathrm{rank} A = 2.$$

$$A^{\mathrm{T}}A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -1 \\ 0 & 17/2 \end{pmatrix} \quad \Rightarrow \mathrm{rank}(A^{\mathrm{T}}A) = 2.$$

$$AA^{\mathrm{T}} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 4 \\ 1 & 4 & 5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 3 & 9/2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \mathrm{rank}(AA^{\mathrm{T}}) = 2.$$

REMARK: It can be shown that $\mathrm{rank}A = \mathrm{rank}(A^{\mathrm{T}}A) = \mathrm{rank}(AA^{\mathrm{T}})$ for any (not necessarily square) matrix A. But it is more subtle than the analyses we have done so far. We will return to this topic in a later lecture, since $A^{\mathrm{T}}A$ is a very important matrix for least-square problems.

Problem 11: Choose three independent columns of $A = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix}$. Then choose a different three independent columns. Explain whether either of these choices forms a basis for $C(A)$.

Solution (10 points)

Method 1: We first need to figure out the dimension of $C(A)$; we can do Gaussian elimination.

$$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So, there are three pivots and hence $\dim C(A) = 3$.

We choose column 1, 2, 4 and put them together as

$$\begin{pmatrix} 2 & 3 & 1 \\ 4 & 12 & 2 \\ 0 & 0 & 9 \\ 0 & 6 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 3 & 1 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \\ 0 & 6 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 3 & 1 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of this matrix is 3 and hence the column space of this matrix is of 3-dimentional. It as to be all of $C(A)$. Hence, columns 1, 2, 4 form a basis of $C(A)$.

We may also choose column 1, 3, 4 and put them together similar as above.

$$\begin{pmatrix} 2 & 4 & 1 \\ 4 & 15 & 2 \\ 0 & 0 & 9 \\ 0 & 7 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 4 & 1 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \\ 0 & 7 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 4 & 1 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \\ 0 & 0 & 0 \end{pmatrix}$$

Same argument as above shows that these three column form a basis of $C(A)$.

<u>Method 2:</u> We first use row operation to turn A into its echelon form.

$$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 2 & 0 & 1/2 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & 1/2 & 0 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1/4 & 0 \\ 0 & 1 & 7/6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $\operatorname{rank} A = 3$ and $N(A)$ is spanned by $x_n = \begin{pmatrix} -1/4 \\ -7/6 \\ 1 \\ 0 \end{pmatrix}$, which gives the relation of the columns. Let v_i denote the column i. Then the special solution x_n gives a relation $-\frac{1}{4}v_1 - \frac{7}{6}v_2 + v_3 = 0$. If we take any two columns from the first three columns and the column 4, they will span a three dimensional space since there will be no relation among them. Hence, they form a basis of $C(A)$.

Problem 12: Find a basis for the space of 2×3 matrices whose nullspace contains $(1, 2, 0)$.

$\boxed{\text{Solution}}$ (10 points)

Method 1: A 2×3 matrix looks like $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$. Since the null-space $N(A)$ contains $(1,2,0)$, we have $a+2b=0$ and $d+2e=0$. Thus the space of 2×3 matrices with the prescribed condition is a subspace of all 2×3 matrices subject to the two equations above. In terms of matrix,

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The free variables are b, c, e, f. We can easily get a basis from special solutions to this new system. Writing in terms of matrix, the basis consists of

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Method 2: The nullspace contains (1,2,0), so the first column should be -2 times the second, and the third column should be anything, hence the matrix should look like $\begin{pmatrix} -2a & a & b \\ -2c & c & d \end{pmatrix}$. There are thus four degrees of freedom (a, b, c, d), thus we expect the space to be four dimensional and the basis to contain four matrices, one for each degree of freedom. For example, $\begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ by setting $a = 1$ and the others to zero; $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ by setting $b = 1$, $\begin{pmatrix} 0 & 0 & 0 \\ -2 & 1 & 0 \end{pmatrix}$ by setting $c = 1$, and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ by setting $d = 1$.

Problem 13: Make the matrix $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$ in Matlab by the command:

```
>> A = [2 1; 6 3]
```

Then compute $b = Ax$ for 100 random x vectors by the command:

```
>> br = A * rand(2, 100);
```

Plot these b vectors as black dots by the commands:

```
>> plot(br(1,:), br(2,:), 'k.')
```

What is the pattern, and why?

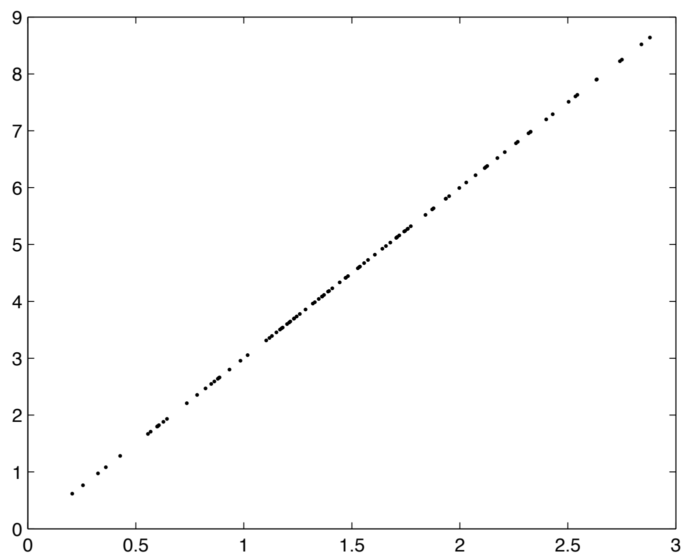
$\boxed{\text{Solution}}$ (10 points)

```
>> A = [2 1; 6 3]

A =

     2     1
     6     3

>> br = A * rand(2, 100);
>> plot(br(1,:), br(2,:), 'k.')
```



Multiplying a vector x on the right means to take the linear combination of the two columns of the matrix A, which gives the column space. We know the column space is the span of the columns, but the first column is twice the first (A has rank 1), so the column space is just the line parallel to $(1, 3)$. What we are plotting is random points in the column space, so they all fall along this line.

REMARK: Moreover, one may notice that the density of dot between $x = 1$ and $x = 2$ is more than away from that. This reflects the absolute value of the two columns. This can be explained by simple probability. Indeed, the probability for $x = x_0$ is the proportional to the length of the interval by slicing the rectangular $0 \le u \le 1, 0 \le v \le 2$ using $u + v = x_0$. That length achieve its maximal when $1 \le x \le 2$.

Students who are interested in this problem are encouraged to discuss with your recitation instructors.