18.06 Problem Set 2

Due Wednesday, 18 February 2008 at 4pm in the undergrad. math office.

1. What threee elimination matrices E_{21}, E_{31}, and E_{32} put A into upper-triangular form $E_{32}E_{31}E_{21}A = U$? Using these, compute the matrix L (and U) to factor $A = LU$.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{pmatrix}$$

2. Suppose we have a 3×3 lower-triangular L matrix of the form

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix}.$$

   (a) When you do the usual Gaussian-elimination steps on L, what matrix do you get?

   (b) When you do the *same* elimination steps to I, what matrix do you get? (Hint: you can write the answer in terms of L very simply.)

   (c) When you apply the same steps to a matrix $A = LU$, what matrix do you get (write your answer in terms of L, U, and/or A).

   (It is possible to answer this question without doing any calculations.)

3. Without computing A or A^{-1} or A^{-2} or A^2 explicitly, compute $A^{-1}x + A^{-2}y$, where you are given the following LU factorization $A = LU$:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad y = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

   (Solve a sequence of triangular systems to get your answer at the end.)

4. Normally, we eliminate downwards to produce an upper-triangular matrix U from a matrix A; suppose we eliminate *upwards* instead to convert A into *lower*-triangular form. (That is, use the last row to produce zeros above the last pivot, the second-to-last row to produce zeros above the second-to-last pivot, and so on.) Do this for the following matrix A, and by doing so find the factors $A = UL$.

$$A = \begin{pmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

5. (a) Write down a permutation matrix P that reverses the order of the rows of a 3×3 matrix. Check that $P^2 = I$.

   (b) Given a lower-triangular matrix L, show how you can multiply (possibly multiple times) by P to get an upper-triangular matrix.

   (c) Multiply this P on both the left and the right of the matrix A from the previous problem to obtain PAP.

   (d) Show how to use your factorization $A = UL$ from the previous problem to get an LU factorization $PAP = L'U'$ where L' and U' are lower- and upper-triangular matrices, respectively. That is, show how to get L' and U' from your answers U and L of the previous problem merely by permutations, with no additional calculation (you do *not* need to re-do the elimination process for PAP). Hint: you can freely insert a factor of $P^2 = I$ where ever you want.

6. Come up with 2×2 matrices A and B, and check by direct calculation that $(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}} \neq A^{\mathrm{T}} B^{\mathrm{T}}$.

7. Express $((AB)^{-1})^{\mathrm{T}}$ in terms of $(A^{-1})^{\mathrm{T}}$ and $(B^{-1})^{\mathrm{T}}$.

8. If L is a lower-triangular matrix, then $(L^{-1})^{\mathrm{T}}$ is __________ triangular.

9. Find a 4×4 permutation matrix P with $P^4 \neq I$.

10. Suppose R is $m \times n$ and $A = A^{\mathrm{T}}$ is a symmetric $m \times m$ matrix.

    (a) Using R^{T}, A, and R, form a new symmetric matrix (transpose it to check that it is symmetric). How many rows and columns does your matrix have?

    (b) Show that $B = R^{\mathrm{T}} R$ has no negative numbers on its diagonal. (Hint: first, explain what vector x gives the i-th diagonal element of B by $b_{ii} = x^{\mathrm{T}} B x$. Then explain why $b_{ii} \geq 0$ for $B = R^{\mathrm{T}} R$.)

11. Suppose $Q^{\mathrm{T}} = Q^{-1}$ for some matrix Q, so that $Q^{\mathrm{T}} Q = I$. Show that the columns of Q are orthogonal unit vectors, i.e. each column q_i has length $\|q_i\|^2 = q_i^{\mathrm{T}} q_i = 1$, and $q_i^{\mathrm{T}} q_j = 0$ for two different columns $i \neq j$.

12. Say whether the following sets of matrices form a subspace of the set of all matrices (under ordinary matrix addition and multiplication by scalars); give a counter-example (something that violates the rules for subspaces) for cases that are *not* a subspace.

    (a) invertible matrices.

    (b) singular matrices

    (c) symmetric matrices ($A = A^{\mathrm{T}}$)

    (d) anti-symmetric matrices ($A = -A^{\mathrm{T}}$)

    (e) unsymmetric matrices ($A \neq A^{\mathrm{T}}$)

13. Find a square matrix A where $C(A^2)$ (the column space of A^2) is smaller than $C(A)$.

14. An $n \times n$ matrix A has $C(A) = \mathbf{R}^n$ if and only if A is a/an __________ matrix.