18.06 Problem Set 2 Solution

Due Wednesday, 18 February 2009 at 4 pm in 2-106.

Total: 155 points.

Problem 1: What threee elimination matrices E_{21}, E_{31}, and E_{32} put A into upper-triangular form $E_{32}E_{31}E_{21}A = U$? Using these, compute the matrix L (and U) to factor $A = LU$.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{pmatrix}$$

$\boxed{\text{Solution}}$ (10 points)

The Gaussian elimination process is as follows:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{pmatrix}.$$

As we can see from this process, the matrix E_{21} corresponds to subtracting twice of the first row from the second row; the matrix E_{31} is trivial; the matrix E_{32} corresponds to subtracting twice of the second row from the third row. In other words,

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$$

Hence,

$$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}; \quad U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{pmatrix}.$$

Problem 2: Suppose we have a 3×3 lower-triangular L matrix of the form

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix}.$$

(a) When you do the usual Gaussian-elimination steps on L, what matrix do you get?

(b) When you do the *same* elimination steps to I, what matrix do you get? (Hint: you can write the answer in terms of L very simply.)

(c) When you apply the same steps to a matrix $A = LU$, what matrix do you get (write your answer in terms of L, U, and/or A).

(It is possible to answer this question without doing any calculations.)

$\boxed{\text{Solution}}$ (15 points = 5+5+5)

(a) We get the identity matrix if we apply the usual Gaussian elimination, because Gaussian elimination puts zeros below the pivots while leaving the pivots (= 1 here) unchanged..

(b) In part (a), we said that doing Gaussian elimination to L gives I—that is, $EL = I$ where E is the product of the elimination matrices (multiplying on the left since these are row operations). But $EL = I$ means that $E = L^{-1}$. Hence, doing the same elimination steps to I gives $EI = E = L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ l_{21}l_{32} - l_{31} & -l_{32} & 1 \end{pmatrix}$.
[Note to grader: the student need not compute L^{-1} explicitly as was done here.

(c) If we do the same elimination steps to $A = LU$, this corresponds to multiplying A on the left by the elimination matrices E, so we get $EA = ELU = (EL)U = U$, using the fact that $EL = I$ from (a). Equivalently, from (b), $E = L^{-1}$ so we get $L^{-1}A = L^{-1}LU = U$.

REMARK: More generally, we applying the same elimination steps to a matrix A, we will get $EA = L^{-1}A$.

Problem 3: Without computing A or A^{-1} or A^{-2} or A^2 explicitly, compute $A^{-1}x + A^{-2}y$, where you are given the following LU factorization $A = LU$:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad y = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

(Solve a sequence of triangular systems to get your answer at the end.)

$\boxed{\text{Solution}}$ (10 points)

The principle of doing this problem is that computing $A^{-1}x$ is equivalent to solving the linear system $Av = x$. The decomposition $A = LU$ helps us to reduce to the triangular system case. Indeed, we first solve the linear system $Lv = x$ and then the system $Uw = v$.

To make the computation easier, write $A^{-1}x + A^{-2}y$ as $A^{-1}(x + A^{-1}y)$.

First, solving $Lv = y$ gives $v_1 = y_1 = -1$, $v_2 = y_2 - v_1 = 2$, and $v_3 = y_3 - v_2 = -1$. Then, solving $Uw = v$ gives $w_3 = -1$, $w_2 = v_2 - w_3 = 3$, and $w_1 = v_1 - w_3 = 0$. So,

$$A^{-1}y = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}, \text{ and } x + A^{-1}y = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}.$$

Now, we solve $Lu = x + A^{-1}y$ and get $u_1 = 1$, $u_2 = 3 - u_1 = 2$, $u_3 = 0 - u_2 = -2$. Finally, we solve $Uz = u$ to get $z_3 = -2$, $z_2 = 2 - z_3 = 4$, and $z_1 = 1 - z_3 = 3$. Hence,

$$A^{-1}x + A^{-2}y = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}.$$

REMARK: When solving a triangular system, if it is upper-triangular, we solve from the bottom to the top; if it is lower-triangular, we solve from the top to the bottom.

Problem 4: Normally, we eliminate downwards to produce an upper-triangular matrix U from a matrix A; suppose we eliminate *upwards* instead to convert A into *lower*-triangular form. (That is, use the last row to produce zeros above the last pivot, the second-to-last row to produce zeros above the second-to-last pivot, and so on.) Do this for the following matrix A, and by doing so find the factors $A = UL$.

$$A = \begin{pmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$\boxed{\text{Solution}}$ (10 points)

The *upwards* Gaussian elimination is as follows:

$$\begin{pmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 4 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

The row operations we used are subtracting the third row from the first and the second, and then subtracting the second row from the first. To sum up this, we have,

$$U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

REMARK: The corresponding U and L in UL decomposition are typically different from the ones obtained in the LU decomposition.

Problem 5:

(a) Write down a permutation matrix P that reverses the order of the rows of a 3×3 matrix. Check that $P^2 = I$.

(b) Given a lower-triangular matrix L, show how you can multiply (possibly multiple times) by P to get an upper-triangular matrix.

(c) Multiply this P on both the left and the right of the matrix A from the previous problem to obtain PAP.

(d) Show how to use your factorization $A = UL$ from the previous problem to get an LU factorization $PAP = L'U'$ where L' and U' are lower- and upper-triangular matrices, respectively. That is, show how to get L' and U' from your answers U and L of the previous problem merely by permutations, with no additional calculation (you do *not* need to re-do the elimination process for PAP). Hint: you can freely insert a factor of $P^2 = I$ where ever you want.

$\boxed{\text{Solution}}$ (20 points = 5+5+5+5)

(a) $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. $P \cdot P = I$ because multiplying P on the left of P is to exchange the first and the third rows of P, which gives the identity matrix.

(b) Let $L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$. First applying P on the left, we exchange the first and the third rows of L, i.e., $PL = \begin{pmatrix} l_{31} & l_{32} & l_{33} \\ l_{21} & l_{22} & 0 \\ l_{11} & 0 & 0 \end{pmatrix}$. Then, applying P on the right of PL, we exchange the first and the third columns of PL, i.e., $PLP = \begin{pmatrix} l_{33} & l_{32} & l_{31} \\ 0 & l_{22} & l_{21} \\ 0 & 0 & l_{11} \end{pmatrix}$.

(c)

$$PAP = P \begin{pmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} P = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 1 \\ 5 & 3 & 1 \end{pmatrix} P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{pmatrix}.$$

(d) $PAP = PULP = PUP^2LP = (PUP)(PLP)$. As we discussed in (b), for a lower-triangular matrix L, PLP is upper-triangular. For the same reason, given a upper-triangular matrix U, PUP becomes lower-triangular. Hence, setting $L' = PUP$ and $U' = PLP$, we have obtained an LU decomposition of PAP by $PAP = L'U'$.

REMARK: The phenomenon in (d) holds for $n \times n$ matrices, with P replaced by the matrix with 1 on the anti-diagonal entries and 0 elsewhere.

Problem 6: Come up with 2×2 matrices A and B, and check by direct calculation that $(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}} \neq A^{\mathrm{T}} B^{\mathrm{T}}$.

$\boxed{\text{Solution}}$ (10 points)

For example, $A = \begin{pmatrix} 1 & 5 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

$$\begin{aligned}
(AB)^{\mathrm{T}} &= \left(\begin{pmatrix} 1 & 5 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right)^{\mathrm{T}} = \begin{pmatrix} 16 & 22 \\ 11 & 16 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 16 & 11 \\ 22 & 16 \end{pmatrix}, \\
B^{\mathrm{T}} A^{\mathrm{T}} &= \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 16 & 11 \\ 22 & 16 \end{pmatrix}, \\
A^{\mathrm{T}} B^{\mathrm{T}} &= \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 11 \\ 11 & 27 \end{pmatrix}.
\end{aligned}$$

Problem 7: Express $((AB)^{-1})^{\mathrm{T}}$ in terms of $(A^{-1})^{\mathrm{T}}$ and $(B^{-1})^{\mathrm{T}}$.

$\boxed{\text{Solution}}$ (5 points)

Since $AB(AB)^{-1} = I$, $(AB)^{-1} = B^{-1}A^{-1}$. Hence, $((AB)^{-1})^{\mathrm{T}} = (B^{-1}A^{-1})^{\mathrm{T}} = (A^{-1})^{\mathrm{T}}(B^{-1})^{\mathrm{T}}$.

Problem 8: If L is a lower-triangular matrix, then $(L^{-1})^{\mathrm{T}}$ is __________ triangular.

$\boxed{\text{Solution}}$ (5 points)

$(L^{-1})^{\mathrm{T}}$ is an upper-triangular matrix. Indeed, L^{-1} is lower-triangular because L is. The transpose carries the upper-triangular matrices to the lower-triangular ones and vice versa.

Problem 9: Find a 4×4 permutation matrix P with $P^4 \neq I$.

$\boxed{\text{Solution}}$ (5 points)

For example, we take the permutation matrix to be the one rotating the first three rows, in other words, $P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then obviously, $P^3 = I$ and hence $P^4 = P \neq I$.

REMARK: It can be shown that if m is the least common multiplier of $1, 2, \dots, n$, then for any permutation matrix P, $P^m = I$. So, in our case $n = 4$ and $m = 12$.

Problem 10: Suppose R is $m \times n$ and $A = A^{\mathrm{T}}$ is a symmetric $m \times m$ matrix.

(a) Using R^{T}, A, and R, form a new symmetric matrix (transpose it to check that it is symmetric). How many rows and columns does your matrix have?

(b) Show that $B = R^{\mathrm{T}}R$ has no negative numbers on its diagonal. (Hint: first, explain what vector x gives the i-th diagonal element of B by $b_{ii} = x^{\mathrm{T}}Bx$. Then explain why $b_{ii} \geq 0$ for $B = R^{\mathrm{T}}R$.)

$\boxed{\text{Solution}}$ (15 points = 5+10)

(a) $R^{\mathrm{T}}AR$ is an $n \times n$ matrix. It is symmetric because $(R^{\mathrm{T}}AR)^{\mathrm{T}} = R^{\mathrm{T}}A^{\mathrm{T}}(R^{\mathrm{T}})^{\mathrm{T}} = R^{\mathrm{T}}AR$.

CAUTION: RAR^{T} is wrong, because we cannot multiply an $m \times n$ matrix from the left to an $m \times m$ matrix.

Also, we can form symmetric matrices $R^{\mathrm{T}}R$ ($m \times m$) and RR^{T} ($n \times n$) without using A. Or, we can use A multiple times by considering $R^{\mathrm{T}}A^2R$, $R^{\mathrm{T}}A^3R$ and so on. These are all symmetric $n \times n$ matrices. (Again, be careful that RA^2R^{T}, RA^3R^{T} and so on are not eligible either.)

(b) There are two (not very different) ways of proving this.

Method 1: The hint suggests a very conceptual way to understand this. If we take x_i to be the vector with 1 on the i-th component and 0 elsewhere, then $b_{ii} = x_i^{\mathrm{T}}Bx_i$ picks up exactly the i-th element on the diagonal. Thus,

$$b_{ii} = x_i^{\mathrm{T}}Bx_i = x_i^{\mathrm{T}}R^{\mathrm{T}}Rx_i = (Rx_i)^{\mathrm{T}}Rx_i = \|\overrightarrow{Rx_i}\|^2 \geq 0.$$

Method 2: We can, alternatively, compute directly.

Denote $R = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{pmatrix}$. Then

$$B = R^{\mathrm{T}}R = \begin{pmatrix} r_{11} & r_{21} & \cdots & r_{m1} \\ r_{12} & r_{22} & \cdots & r_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ r_{1n} & r_{2n} & \cdots & r_{nm} \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{pmatrix}$$

Hence, For the i-th element b_{ii} on the diagonal, we have

$$b_{ii} = \sum_{j=1}^{n} r_{ji}r_{ji} = \sum_{j=1}^{n} r_{ji}^2 \geq 0.$$

Problem 11: Suppose $Q^{\mathrm{T}} = Q^{-1}$ for some matrix Q, so that $Q^{\mathrm{T}}Q = I$. Show that the columns of Q are orthogonal unit vectors, i.e. each column q_i has length $\|q_i\|^2 = q_i^{\mathrm{T}}q_i = 1$, and $q_i^{\mathrm{T}}q_j = 0$ for two different columns $i \neq j$.

$\boxed{\text{Solution}}$ (10 points)

First of all, the matrix Q is a square matrix, say $n \times n$. Again, we give two proofs, based on the two methods from the previous problem, respectively.

Method 1: This is more conceptual than the second proof.

Let x_i denote the vector with 1 on the i-th component and 0 elsewhere. Then i-th column of Q is $q_i = Qx_i$. Hence,

$$q_i^{\mathrm{T}} q_j = x_i^{\mathrm{T}} Q^{\mathrm{T}} Q x_j = x_i^{\mathrm{T}} I x_j = x_i^{\mathrm{T}} x_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Method 2: We may also choose to compute directly, which is none other than writing everything explicit from the previous proof.

Denote $Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$. Then the i-th column is $q_i = \begin{pmatrix} q_{i1} \\ q_{i2} \\ \vdots \\ q_{in} \end{pmatrix}$. Note that

$$Q^{\mathrm{T}} Q = \begin{pmatrix} q_{11} & q_{21} & \cdots & q_{n1} \\ q_{12} & q_{22} & \cdots & q_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ q_{1n} & q_{2n} & \cdots & q_{nn} \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix} = \begin{pmatrix} \sum_a q_{1a} q_{a1} & \sum_a q_{2a} q_{a1} & \cdots & \sum_a q_{na} q_{a1} \\ \sum_j q_{1a} q_{a2} & \sum_a q_{2a} q_{a2} & \cdots & \sum_a q_{na} q_{a2} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_a q_{1a} q_{an} & \sum_a q_{2a} q_{an} & \cdots & \sum_a q_{na} q_{an} \end{pmatrix}$$

Hence,

$$\sum_a q_{ia} q_{aj} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$q_i^{\mathrm{T}} q_j = \begin{pmatrix} q_{i1} & q_{i2} & \cdots & q_{in} \end{pmatrix} \begin{pmatrix} q_{j1} \\ q_{j2} \\ \vdots \\ q_{jn} \end{pmatrix} = \sum_a q_{ia} q_{aj} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Problem 12: Say whether the following sets of matrices form a subspace of the set of all matrices (under ordinary matrix addition and multiplication by scalars); give a counter-example (something that violates the rules for subspaces) for cases that are *not* a subspace.

(a) invertible matrices

(b) singular matrices

(c) symmetric matrices ($A = A^{\mathrm{T}}$)

(d) anti-symmetric matrices ($A = -A^{\mathrm{T}}$)

(e) unsymmetric matrices ($A \neq A^{\mathrm{T}}$)

$\boxed{\text{Solution}}$ (25 points = 5+5+5+5+5)

(a) No. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ are invertible matrices but $A+B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ is not invertible. Actually, more importantly, the zero matrix is *not* invertible but any linear subspace should contain the zero matrix. So invertible matrices do not form a linear subspace.

(b) No. For example, $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are singular matrices but $A + B = I$ is not singular.

(c) Yes. This is because the multiples of symmetric matrices and sums of symmetric matrices are still symmetric. In other words, if A and B are symmetric matrices, $(aA)^{\mathrm{T}} = aA^{\mathrm{T}} = aA \Rightarrow aA$ is symmetric for $a \in \mathbb{R}$, and $(A+B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}} = A + B \Rightarrow A + B$ is symmetric.

(d) Yes. The argument is similar to (c) but with a negative sign everywhere when taking away the transpose symbol.

(e) No, because the zero matrix is not unsymmetric.

Problem 13: Find a square matrix A where $C(A^2)$ (the column space of A^2) is smaller than $C(A)$.

$\boxed{\text{Solution}}$ (5 points)

For example, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus, $C(A)$ is generated by vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is of one dimensional, but $C(A^2)$ is the zero space. Hence, $C(A^2)$ is strictly smaller than $C(A)$.

Problem 14: An $n \times n$ matrix A has $C(A) = \mathbf{R}^n$ if and only if A is a/an __________ matrix.

$\boxed{\text{Solution}}$ (5 points)

invertible.

$C(A) = \mathbb{R}^n$ if and only if for all vector $b \in \mathbb{R}^n$, we can find at least one solution v for $Av = b$. This is in turn equivalent to A being an invertible matrix.