18.06 Spring 2009 Exam 3 Practice

General comments

Exam 2 covers the first 31 lectures of 18.06, mainly focusing on lectures 19–31 (eigenproblems). The topics covered are (very briefly summarized):

1. All of the topics from exams 1 and 2, although of course these are not the focus of the exam.

2. Determinants: their properties, how to compute them (simple formulas for 2×2 and 3×3, usually by elimination for matrices $> 3 \times 3$), their relationship to linear equations (zero determinant = singular), their use for eigenvalue problems.

3. Eigenvalues and eigenvectors: their definition $\boxed{A\vec{x} = \lambda\vec{x}}$, their properties, the fact that for an eigenvector the matrix (or *any function of the matrix*) acts just like a number. Computing λ from the characteristic polynomial $\det(A - \lambda I)$ and $\vec{x}$ from $N(A - \lambda I)$; zero eigenvalues $\lambda = 0$ just correspond to $N(A)$. Understand (from the definition) why, if A has an eigenvalue λ, then A^k has an eigenvalue λ^k, αA has an eigenvalue $\alpha\lambda$, and $A + \beta I$ has an eigenvalue $\lambda + \beta$, all with the *same* eigenvector.

4. Diagonalization $A = S\Lambda S^{-1}$: where it comes from, its use in understanding properties of matrices and eigenvalues. **The basic idea that, to solve a problem involving A, you first expand your vector in the basis of the eigenvectors (S), then for each eigenvector you treat A as just a number λ, then at the end you add up the solutions.**

5. Similar matrices: A and $B = MAM^{-1}$ have the same eigenvalues for any invertible matrix M, and if $A\vec{x} = \lambda\vec{x}$ then $B\vec{y} = \lambda\vec{y}$ for $\vec{y} = M\vec{x}$. Similar matrices have the same trace (sum of the eigenvalues) and determinant (product of the eigenvalues).

6. Using eigenvalues/eigenvectors to solve problems involving matrix powers, such as linear recurrences (e.g. Fibonacci). Multiplying by A many times tends towards the eigenvector for the largest $|\lambda|$. Markov matrices: what the defining properties are, and the consequences (a steady state with $\lambda = 1$, all other solutions decay away, the sum of the components of the vector is conserved, a unique steady state if all entries of the matrix are > 0). $A^n = S\Lambda^n S^{-1}$.

7. Using eigenvalues/eigenvectors to solve linear systems of differential equations $\frac{d\vec{u}}{dt} = A\vec{u}$ with initial conditions $\vec{u}(0)$. Practical scheme: expand $\vec{u}(0)$ in eigenvector basis and multiply each term by $e^{\lambda t}$. Formal solution: $e^{At}\vec{u}(0)$, and the meaning of the matrix exponential $e^A = Se^{\Lambda}S^{-1}$ and how to compute it and manipulate it.

8. If $A = A^T$ (real-symmetric), then the eigenvalues are real and the eigenvectors are orthogonal (or can be chosen orthogonal), and A is diagonalizable as $A = Q\Lambda Q^T$ for an orthogonal Q. If $A = B^T B$ where B has full column rank, then A is positive definite: all $\lambda > 0$ and all pivots > 0 and $\vec{y}^T A\vec{y} > 0$ for any $\vec{y} \neq 0$; connection to minimization problems (like least-squares).

9. Complex matrices, for which we replace $\vec{x}^T$ and A^T by and $\vec{x}^H = \overline{\vec{x}^T}$ $A^H = \overline{A^T}$ (and why). What to do if you get a complex λ: consequences for matrix powers (recurrence relations) and differential equations are oscillating solutions, using $e^{i\theta} = \cos\theta + i\sin\theta$.

10. Defective matrices and generalized eigenvectors: what to do if A is not diagonalizable, especially for a practical problem like $A^k\vec{u}$ or $e^{At}\vec{u}$. (Note that real-symmetric, real-orthogonal, Hermitian, and unitary matrices are never defective, nor are $n \times n$ matrices with n distinct eigenvalues.)

11. Singular value decomposition $A = U\Sigma V^T$ and their relationship to eigenvectors/eigenvalues of A^TA and AA^T.

The central concept from this part of the course is highlighted in boldface above. Once you have an eigenvector, *any* operation involving the matrix just becomes that operation with the single number λ. And single numbers are easy to handle. So, we try to find the eigenvectors and then express every vector in that basis (aside from rare defective cases), at which point problems become easy (ideally)! Also, you should be able to recognize and reason about how and why special forms of the matrix A (symmetric, Markov, singular, etcetera) give you additional information about the eigenvectors and eigenvalues.

Defective matrices and SVDs will see at most limited coverage on the exam, perhaps one part of a problem each, at most.

Some practice problems

The 18.06 web site has exams from previous terms that you can download, with solutions. I've listed a few practice exam problems that I like below, but there are plenty more to choose from. The exam will consist of 3 or 4 questions (perhaps with several parts each), and you will have one hour. You can find the solutions to these problems on the 18.06 web site (in the section for old exams/psets).

1. (Fall 2002 exam 3.) **(a)** What are the eigenvalues of the 5×5 matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$? Please look at A, not at $\det(A - \lambda I)$. **(b)** Solve $\frac{d\vec{u}}{dt} = A\vec{u}$ starting from $\vec{u}(0) = (0,1,1,1,2)^T$. (First split $\vec{u}(0)$ as the sum of two eigenvectors of A.) **(c)** Using part (a), what are the *eigenvalues* and *trace* and *determinant* of the matrix B which is the same as A except that it has zeros on its diagonal?

2. (Fall 2002 exam 3.) **(a)** if A is similar to B show that e^A is similar to e^B. (Hint: first write down the definitions of "similar" and e^A.) **(b)** If A has 3 eigenvalues $\lambda = 0, 2, 4$, find the eigenvalues of e^A. **(c)** Explain this connection with determinants: $\det(e^A) = e^{\text{trace of } A}$.

3. (Fall 2002 exam 3.) Companies in the US, Asia, and Europe have assets of \$12 trillion. At the start, \$6 trillion are in the US and \$6 trillion are in Europe. Each year, half the US money stays home, $1/4$ each goes to Asia and Europe. For Asia and Europe, half stays home and half is sent to the US, hence

$$\begin{pmatrix} \text{US} \\ \text{Asia} \\ \text{Europe} \end{pmatrix}_{\text{year } k+1} = \begin{pmatrix} .5 & .5 & .5 \\ .25 & .5 & 0 \\ .25 & 0 & .5 \end{pmatrix} \begin{pmatrix} \text{US} \\ \text{Asia} \\ \text{Europe} \end{pmatrix}_{\text{year } k+1}.$$

(a) The eigenvalues and eigenvectors of this *singular* matrix A are what? **(b)** The limiting distribution of the \$12 trillion after many many years is US=?, Asia=?, Europe=?

4. (Fall 2002 exam 2.) If you know that $\det A = 6$, what is $\det B$ for B given by:

$$A = \begin{pmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{pmatrix} \qquad B = \begin{pmatrix} \text{row 3 + row 2 + row 1} \\ \text{row 2 + row 1} \\ \text{row 1} \end{pmatrix}$$

5. (Spring 2004 exam 3.) For the symmetric matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$, you are given that one of the eigenvalues is $\lambda = 1$ with a line of eigenvectors $\vec{x} = (c, c, 0)$. **(a)** That line is the nullspace of what matrix constructed from A? **(b)** Find (in any way) the other two eigenvalues of A and two corresponding eigenvectors. **(c)** The diagonalization $A = S\Lambda S^{-1}$ has an especially nice form because $A = A^T$. Write all entries in the nice symmetric diagonalization of A. **(d)** Give a reason why e^A *is* or *is not* a symmetric positive-definite matrix.

6. (Spring 2004 exam 3.) **(a)** Find the eigenvalues and eigenvectors (depending on c) of $A = \begin{pmatrix} 0.3 & c \\ 0.7 & 1-c \end{pmatrix}$. For which c is the matrix A *not diagonalizable*?[1] **(b)** What is the largest range of (real) values of c so that A^n approaches a limiting matrix A^∞ as $n \to \infty$? **(c)** What is that limit of A^n (still depending on c)?

7. (Spring 2005 exam 3.) **(a)** Find all the eigenvalues and all the eigenvectors of the following A. *It is a symmetric Markov matrix with a repeated eigenvalue.*

$$A = \begin{pmatrix} 2/4 & 1/4 & 1/4 \\ 1/4 & 2/4 & 1/4 \\ 1/4 & 1/4 & 2/4 \end{pmatrix}.$$

(b) Find the limit of A^k as $k \to \infty$. **(c)** Choose any positive numbers r, s, and t so that $A - rI$ is positive-definite, $A - sI$ is indefinite, and $A - tI$ is negative definite. **(d)** Suppose that this $A = B^T B$. What are the singular values σ_i in the SVD of B?

8. (Spring 2005 exam 3.) **(a)** Complete this 2×2 matrix A, depending on the real number a, so that its eigenvalues are $\lambda = 1$ and $\lambda = -1$. $A = \begin{pmatrix} a & 1 \\ ? & ? \end{pmatrix}$. **(b)** How do you know that A has two independent eigenvectors? **(c)** Which choices of a give orthogonal eigenvectors and which don't?

9. (Spring 2005 exam 3.) Suppose that the 3×3 matrix A has 3 independent eigenvectors $\vec{x}_{1,2,3}$ and corresponding eigenvalues $\lambda_{1,2,3}$. (The λ's might not be different.) **(a)** Describe the general form of every solution $\vec{u}(t)$to the differential equation $\frac{d\vec{u}}{dt} = A\vec{u}$ in terms of the λ's and $\vec{x}$'s. (The answer $e^{At}\vec{u}(0)$ is not sufficient.) **(b)** Starting from any vector $\vec{u}_0$, suppose $\vec{u}_{k+1} = A\vec{u}_k$. What are the conditions on the $\vec{x}$'s and λ's to guarantee that $\vec{u}_k \to 0$ as $k \to \infty$? Why?

10. (Fall 2005 exam 3.) This 4×4 matrix H is a special matrix called a "Hadamard matrix:"

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

It has two key properties: $H^T = H$, and $H^2 = 4I$. **(a)** Figure out the eigenvalues of H and explain your reasoning. **(b)** Figure out H^{-1} and $\det H$. Explain your reasoning. **(c)** This matrix S contains three eigenvectors of H. Find a 4-th eigenvector $\vec{x}_4$ and explain your reasoning.

$$S = \begin{pmatrix} 1 & 1 & 0 & ? \\ 1 & 0 & -1 & ? \\ 1 & 0 & 1 & ? \\ -1 & 1 & 0 & ? \end{pmatrix}.$$

e**(d)** Find the solution to $d\vec{u}/dt = H\vec{u}$ given that $\vec{u}(0)$ is the 3rd column of S.

11. (Fall 2005 exam 3.) Suppose A is a 3×3 symmetric matrix with eigenvalues $2, 5, 7$ and corresponding eigenvectors $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$. **(a)** Suppose $\vec{x}$ is a linear combination $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$. Find $A\vec{x}$. Now find $\vec{x}^T A\vec{x}$ using the symmetry of A. Explain why $\vec{x}^T A\vec{x} > 0$ unless $\vec{x} = 0$.

12. (Fall 2006 exam 3.) **(a)** Find all three eigenvalues of A, and an eigenvector matrix S. $A = \begin{pmatrix} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{pmatrix}$. **(b)** Explain why $A^{1001} = A$. Is $A^{1000} = I$? **(c)** The matrix $A^T A$ for this A is $A^T A = \begin{pmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ -4 & 8 & 42 \end{pmatrix}$.How many

[1] The solutions are a little tricky. A is *not* a Markov matrix because c may be < 0. However, its columns sum to 1, and that was enough to give us an eigenvalue $\lambda = 1$ in our analysis of Markov matrices. In the non-diagonalizable case, the solution's formula for part **(b)** is incorrect. As we know from lecture 30, for a repeated eigenvalue $\lambda = 1$ that is defective, there is a term in A^n that goes as 1^n and another term that goes as $n\,1^{n-1}$. Since the latter blows up, the defective case does not have a finite A^∞ limit.

eigenvalues of A^TA are positive? zero? negative? Does A^TA have the same eigenvectors as A? (Don't compute anything, but explain your answers.)

13. (Fall 2006 exam 3.) Suppose the $n \times n$ matrix A has n orthonormal eigenvectors $\vec{q}_1, \ldots, \vec{q}_n$ and n positive eigenvalues $\lambda_1, \ldots, \lambda_n$. That is, $A\vec{q}_j = \lambda_j \vec{q}_j$. **(a)** What are the eigenvalues and eigenvectors of A^{-1}? **(b)** Any vector $\vec{b}$ can be written as a combination of the eigenvectors $\vec{b} = c_1\vec{q}_1 + c_2\vec{q}_2 + \cdots + c_n\vec{q}_n$ for some coefficients c_j. What is a quick formula for c_1 using the orthogonality of the $\vec{q}$'s? **(c)** The solution to $A\vec{x} = \vec{b}$ is also a combination of the eigenvectors $A^{-1}\vec{b} = d_1\vec{q}_1 + d_2\vec{q}_2 + \cdots + d_n\vec{q}_n$. What is a quick formula for d_1. (You can write it in terms of the c's even if you didn't answer part b.)

14. (Fall 2007 exam 3.) Suppose that we form a sequence of real numbers f_k defined by the recurrence relation $f_{k+1} = f_k - f_{k-1} + f_{k-2}$, starting with the initial numbers $f_0 = 2$, $f_1 = 1$, and $f_2 = 2$. **(a)** Define a 3-component vector $\vec{g}_k = (f_k, f_{k-1}, f_{k-2})$ and a 3×3 matrix A so that $\vec{g}_{k+1} = A\vec{g}_k$. **(b)** If you constructed A correctly, the three eigenvalues should be 1 and $\pm i$, and the latter two eigenvectors should be $(-1, \pm i, 1)$. Check that you have these $\pm i$ eigenvalues and eigenvectors, and find the $\lambda = 1$ eigenvector. **(c)** Give an explicit formula for f_k for any k (formulas involving A^k are not acceptable; elementary arithmetic and powers of complex numbers only). **(d)** Is there any choice of initial conditions (f_0, f_1, and f_2) that will make $|f_k|$ diverge as $k \to \infty$? Explain.

15. (Fall 2007 exam 3.) Some 3×3 real matrix A has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 2$, with corresponding eigenvectors $\vec{x}_1 = (1,0,0)$, $\vec{x}_2 = (0,1,2)$, and $\vec{x}_3 = (0,1,1)$. **(a)** Give a basis for the nullspace $N(A)$, the column space $C(A)$, and the row space $C(A^T)$. **(b)** Find *all* solutions (the complete solution) $\vec{x}$ to $A\vec{x} = \vec{x}_2 - 3\vec{x}_3$. **(c)** Is A real-symmetic, orthogonal, Markov, or none of the above?