18.06 Spring 2009 Exam 2 Practice

General comments

Exam 2 covers the first 18 lectures of 18.06. It does *not* cover determinants (lectures 19 and 20). There will also be *no* questions on graphs and networks. The topics covered are (very briefly summarized):

1. All of the topics from exam 1.

2. Linear independence [key point: the columns of a matrix A are independent if $N(A) = \{0\}$], bases (an independent set of vectors that spans a space), and dimension of subspaces (the number of vectors in *any* basis).

3. The four fundamental subspaces (key points: their dimensions for a given rank r and $m \times n$ matrix A, their relationship to the solutions [if any] of $Ax = b$, their orthogonal complements, and how/why we can find bases for them via the elimination process).

4. What happens to the four subspaces as we do matrix operations, especially elimination steps and more generally how the subspaces of AB compare to those of A and B. The fact (important for projection and least-squares!) that $A^T A$ has the same rank as A, the same null space as A, and the same column space as A^T, and why (we proved this in class and another way in homework).

5. Orthogonal complements $S^\perp$ for subspaces S, especially (but not only) the four fundamental subspaces.

6. Orthogonal projections: given a matrix A, the projection of b onto $C(A)$ is $p = A\hat{x}$ where $\hat{x}$ solves $A^T A\hat{x} = A^T b$ [always solvable since $C(A^T A) = C(A^T)$]. If A has full column rank, then $A^T A$ is invertible and we can write the projection matrix $P = A(A^T A)^{-1} A^T$ (so that $A\hat{x} = Pb$, but it is *much* quicker to solve $A^T A\hat{x} = A^T b$ by elimination than to compute P in general). $e = b - A\hat{x}$ is in $C(A)^\perp = N(A^T)$, and $I - P$ is the projection matrix onto $N(A^T)$.

7. Least-squares: $\hat{x}$ minimizes $\|Ax - b\|^2$ over all x, and is the *least-squares* solution. That is, $p = A\hat{x}$ is the *closest* point to b in $C(A)$. Application to least-square curve fitting, minimizing the sum of the squares of the errors.

8. Orthonormal bases, forming the columns of a matrix Q with $Q^T Q = I$. The projection matrix onto $C(Q)$ is just QQ^T, and $\hat{x} = Q^T b$. Obtaining Q from A (i.e., an orthonormal basis from any basis) by Gram-Schmidt, and the correspondence of this process to $A = QR$ factorization where $R = Q^T A$ is invertible and upper-triangular. Using $A = QR$ to solve equations (either $Ax = b$ or $A^T A\hat{x} = A^T b$). Q is an *orthogonal matrix* only if it is square, in which case $Q^T = Q^{-1}$.

9. Dot products of functions, and hence Gram-Schmidt, orthonormal bases (e.g. Fourier series or orthogonal polynomials), orthogonal projection, and least-squares for functions.

As usual, the exam questions may turn these concepts around a bit, e.g. giving the answer and asking you to work backwards towards the question, or ask about the same concept in a slightly changed context. We want to know that you have really internalized these concepts, not just memorizing an algorithm but knowing *why* the method works and where it came from.

Some practice problems

The 18.06 web site has exams from previous terms that you can download, with solutions. I've listed a few practice exam problems that I like below, but there are plenty more to choose from. (Note: exam 2 in several previous terms asked about determinants; we *won't* have any determinant questions until exam 3.) The exam will consist of 3 or 4 questions (perhaps with several parts each), and you will have one hour. You can find the solutions to these problems on the 18.06 web site (in the section for old exams/psets). On the last page I give practice problems for orthogonal functions and orthogonal projections of functions.

1. (Fall 2002 exam 2.) **(a)** Choose c and the last column of Q so that you have an orthogonal matrix:

$$Q = c \begin{bmatrix} 1 & -1 & -1 & ? \\ -1 & 1 & -1 & ? \\ -1 & -1 & -1 & ? \\ -1 & -1 & 1 & ? \end{bmatrix}.$$

   (b) Project $b = (1,1,1,1)^T$ onto the first column of Q. Then project b onto the plane spanned by the first two columns. **(c)** Suppose the last column of this matrix (where the ?'s are) were changed to $(1,1,1,1)^T$. Call this new matrix A. If Gram-Schmidt is applied to the 4 columns of A, what would be the 4 outputs q_1, q_2, q_3, q_4? (Don't do a lot of calculations...please!)

2. (Fall 2008 exam 2.) [The parts of this question are independent and can be done in any order.] **(a)** P is the projection matrix onto $C(A)$, where A has independent columns. Q is a square orthogonal matrix with the same number of rows as A. In its simplest form, in terms of P and Q, what is the projection matrix onto the column space of QA? **(b)** The vectors a, b, and c are independent. The matrix P is the projection matrix onto the span of a and b. Suppose we apply Gram-Schmidt onto the vectors a, b,and c to produce orthonormal vectors q_1,q_2,and q_3. Write the unit vector q_3 in simplest form in terms of P and c only. **(c)** The vectors a, b, and c are independent, and the matrix A has these three vectors as its columns. You are given the QR decomposition of A, where Q is orthogonal and R is 3×3 upper-triangular as usual. Write $\|c\|$ in terms of only the elements of R, in simplest form.

3. (Fall 2008 exam 2.) Suppose we have obtained from measurements n data points (t_i, b_i) and you are asked to find a best least-squares fit function of the form $y = C + Dt + E(1-t)$. Are C, D, and E uniquely determined? Write down a solvable system of equations that gives a solution to the least-squares problem.

4. (Fall 2008 exam 2.) **(a)** If A is invertible, must the column space of A^{-1} be the same as the column space of A? **(b)** If A is square, must the column space of A^2 be the same as the column space of A?

5. (Fall 2005 exam 1.) Suppose A is $m \times n$ with *linearly dependent columns*. Complete with as much true information as possible: **(a)** The rank of A is .....? **(b)** The nullspace of A contains .....? **(c)** The equation $A^T y = b$ has no solution for some right-hand sides b because ......? (more words needed)

6. (Fall 2005 exam 1.) Suppose A is the 3×4 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}.$$

   (a) A basis for $C(A)$ is .....? **(b)** For which vectors $b = (b_1, b_2, b_3)^T$ does $Ax = b$ have a solution? (Give specific conditions on $b_{1,2,3}$.) **(c)** Explain why there is no 4×3 matrix B for which $AB = I$ (3×3). Give a good reason (the mere fact that A is rectangular is *not* sufficient).

7. (Spring 2005 exam 1.) Suppose the columns of a 7×4 matrix A are linearly independent. **(a)** After row operations reduce A to U or R, how many rows will be all zero (or is it impossible to tell)? **(b)** Assume that no row swaps were required for elimination. What is the row space of A? Explain why this equation will surely be solvable: $A^T y = (1,0,0,0)^T$.

8. (Fall 2005 exam 2.) The matrix Q has orthonormal columns q_1, q_2, q_3:

$$Q = \begin{bmatrix} 0.1 & 0.5 & a \\ 0.7 & 0.5 & b \\ 0.1 & -0.5 & c \\ 0.7 & -0.5 & d \end{bmatrix}.$$

(a) What equations must be satisfied by the numbers a, b, c, d? Is there a unique choice for those (real) numbers, apart from multiplying them all by -1? **(c)** Suppose Gram-Schmidt starts with those same first two columns and with the third column $a_3 = (1,1,1,1)^T$. What third column would it choose for q_3. (You can leave a square root as $\sqrt{\cdots}$ if you want to.)

9. (Fall 2005 exam 2.) Our measurements at times $t = 1,2,3$ are $b = 1,4,b_3$. We want to fit those points by the nearest line $C + Dt$, using least-squares. **(a)** Which value for b_3 will put the three points on a straight line? Give C and D for this line. Will least squares choose that line if the third measurement is $b_3 = 9$? (Yes or no.) **(b)** What is the lienar system $Ax = b$ that would be solved exactly for $x = (C,D)$ if the three points do lie on a line? Compute the projection matrix P onto the column space of A. You can use the 2×2 inverse formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. **(c)** What is the rank of that projection matrix P? How is the column space of P related to the column space of A? (You can answer this part without your answer from **b**.) **(d)** Suppose $b_3 = 1$. Write down the equation for the best least-squares solution $\hat{x}$, and show that the best straight line is horizontal in this case.

10. (Fall 2006 exam 2.) Suppose we take measurements at the 21 equally spaced times $t = -10, -9, \ldots, 9, 10$. All measurements are $b_i = 0$ except that $b_{11} = 1$ at the middle time $t = 0$. **(a)** Using least squares, what are the best C and D to fit those 21 points by a straight line $C + Dt$? **(b)** You are projecting the vector b onto what subspace? (*Give a basis.*) Find a nonzero vector perpendicular to that subspace.

11. (Fall 2006 exam 2.) The Gram-Schmidt methods produces orthonormal vectors q_1, q_2, q_3 from independent vectors a_1, a_2, a_3 in $\mathbb{R}^5$. Put those vectors into the columns of 5×3 matrices Q and A, respectively. **(a)** Give formulas using Q and A for the projection matrices P_Q and P_A onto the column spaces of Q and A, respectively. **(b)** Does P_Q equal P_A, and why or why not? What is $P_Q Q$? **(c)** Suppose a_4 is a new vector, and a_1, a_2, a_3, a_4 are independent. Which of the following (if any) is the new Gram-Schmidt vector q_4? **1:** $\frac{P_Q a_4}{\|P_Q a_4\|}$. **2:** $\frac{a_4 - \frac{a_4^T a_1}{a_1^T a_1} a_1 - \frac{a_4^T a_2}{a_2^T a_2} a_2 - \frac{a_4^T a_3}{a_3^T a_3} a_3}{\|\cdots \text{same vector} \cdots\|}$. **3.** $\frac{a_4 - P_A a_4}{\|a_4 - P_A a_4\|}$.

12. (Spring 2004 exam 2.) We are given two vectors a and b in $\mathbb{R}^4$, $a = (2,5,2,4)^T$ and $b = (1,2,1,0)^T$. **(a)** Find the projection p of the vector b onto the line through a. Check(!) that the error $e = b - p$ is perpendicular to....what? **(b)** The subspace S of all vectors in $\mathbb{R}^4$ that are perpendicular to this a is 3-dimensional. Compute the projection q of b onto this perpendicular subspace S. (It doesn't need a big computation!)

13. (Spring 2004 exam 2.) Suppose that q_1, q_2, and q_3 are 3 orthonormal vectors in $\mathbb{R}^n$. They go into the columns of an $n \times 3$ matrix Q. **(a)** What inequality ($\leq$ or $\geq$) do you know for n? Is there any condition on n required in order to have $Q^T Q = I$? Is there any condition on n required to have $QQ^T = I$? **(b)** Give a nice matrix formula involving b and Q for the projection p of a vector b onto the column space of Q. Complete the sentence: p is the closest vector .... **(c)** Suppose the projection of b onto that column space is $p = c_1 q_1 + c_2 q_2 + c_3 q_3$. Find a formula for c_1 that only involves b and q_1 (possibly using dot products).

14. (Spring 2005 exam 2.) If the output vectors from Gram-Schmidt are: $q_1 = (\cos\theta, \sin\theta)^T$ and $q_2 = (-\sin\theta, \cos\theta)$ for some θ, describe all possible input vectors a_1 and a_2.

15. (Spring 2005 exam 2.) If a and b are nonzero vectors in $\mathbb{R}^n$, what number x minimizes the squared length $\|b - xa\|^2$?

16. (Spring 2005 exam 2.) Find the projection p of the vector $b = (1,2,6)^T$ onto the plane $x + y + z = 0$ in $\mathbb{R}^3$. (You may want to first find a basis for this 2-dimensional subspace, perhaps even an orthogonal basis.)

17. (Spring 2005 exam 2.) You are given the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Suppose P_1 is the projection matrix onto the 1-dimensional subspace spanned by the first column of A. Suppose P_2 is the projection matrix onto the 2-dimensional column space of A. After thinking a little, compute the product $P_2 P_1$.

None of the first two exams in previous terms covered orthogonal functions—these are in the standard 18.06 syllabus, especially Fourier series, but previously weren't covered until later in the term, after eigenproblems. A couple of problems about orthogonal functions appeared on your last problem set, which you should review, and a couple more practice problems on this topic are:

1. Suppose you are given three functions $a_1(t)$, $a_2(t)$, and $b(t)$ for $0 \le t \le 1$. Define dot products of any two functions $f(t)$ and $g(t)$ by $f(t) \cdot g(t) = \int_0^1 f(t)g(t)dt$ (hence, $\|f(t)\| = \sqrt{\int_0^1 f(t)^2 dt}$). Suppose we want the "best-fit" function $p(t) = Ca_1(t) + Da_2(t)$ that minimizes $\|p(t) - b(t)\|$ over all possible C and D. Give an explicit formula for $p(t)$ in terms of some integrals and other expressions involving $a_1(t)$, $a_2(t)$, and $b(t)$ only.

2. The functions $q_1(t) = \sin(t)/\sqrt{\pi}$, $q_2(t) = \sin(2t)/\sqrt{\pi}$, and $q_3(t) = \cos(t)/\sqrt{\pi}$ are orthonormal if we define dot products of any two functions $f(t)$ and $g(t)$ by $f(t) \cdot g(t) = \int_0^{2\pi} f(t)g(t)dt$. **(a)** Write the function $b(t) = t$ as the sum of two functions, one in the span of q_1, q_2 and q_3 and one perpendicular to q_1, q_2 and q_3. You should write your answer explicitly in terms of integrals *etc.*, but you need not evaluate the integrals (this isn't 18.01). **(b)** If you were to do Gram-Schmidt on the set of four functions q_1, q_2, q_3, b, in that order, what would you get?

Solutions:

1. This is just a least-squares problem. There are a couple of ways to do this, but the way we learned in class is to first find an orthonormal basis by Gram-Schmidt: $q_1(t) = a_1/\|a_1\| = a_1(t)/\sqrt{\int_0^1 a_1(t)^2 dt}$, $q_2(t) = (a_2 - q_1[q_1 \cdot a_2])/\|\cdots\| = [a_2 - q_1 \int_0^1 q_1(t)a_2(t)dt]/\|\cdots\|$. Then $p(t) = q_1(q_1 \cdot b) + q_2(q_2 \cdot b) = q_1(t)\int_0^1 q_1(t')b(t')dt' + q_2(t')\int_0^1 q_2(t')b(t')dt'$.

2. **(a)** We are just writing $b(t) = p(t) + e(t)$, where $p(t)$ is the orthogonal projection and $e(t) = b(t) - p(t)$. Exactly as for vectors, we can write the orthogonal projection as:

$$p(t) = \sum_{i=1}^{3} q_i(q_i \cdot b) = \frac{\sin(t)}{\pi}\int_0^{2\pi} \sin(t')t'dt' + \frac{\sin(2t)}{\pi}\int_0^{2\pi} \sin(2t')t'dt' + \frac{\cos(t)}{\pi}\int_0^{2\pi} \cos(t')t'dt',$$

and thus $e(t) = t - p(t)$ is perpendicular to q_1, q_2, q_3. **(b)** q_1 to q_3 are already orthonormal, so they wouldn't be changed by Gram-Schmidt. When you do Gram-Schmidt on the last function $b(t)$, you would subtract off the projection and then normalize...but this is precisely the function $q_4(t) = e(t)/\|e(t)\| = e(t)/\sqrt{\int_0^{2\pi} e(t')^2 dt'}$.

The key point that I want you to understand is that you just do exactly the same steps as you would for vectors, and the "only" change is that the dot products become some kind of integral (depending on what the function dot product was chosen to be).