# 18.06 Spring 2009 Exam 1 Practice

## General comments

Exam 1 covers the first 8 lectures of 18.06:

1. The Geometry of Linear Equations
2. Elimination and Matrix Operations (elimination, pivots, etcetera; different viewpoints of $AB$ and $Ax$ and $x^T A$, e.g. as linear combinations of rows or columns)
3. Elimination Matrices and Matrix Inverses (row operations = multiplying on left by elimination matrices, Gauss-Jordan elimination and what happens when you repeat the elimination steps on $I$)
4. $A = LU$ Factorization (for example, the relationship between $L$ and the elimination steps, and solving problems with $A$ in terms of the triangular matrices $L$ and $U$)
5. Permutations, Dot Products, and Transposes (relationship between dot products and transposes, $(AB)^T = B^T A^T$, permutation matrices, etcetera)
6. Vector Spaces and Subspaces (for example, the column space and nullspace, what is and isn't a subspace in general, and other vector spaces/subspaces e.g. using matrices and functions)
7. Solving $Ax = 0$ (the nullspace), echelon form $U$, row-reduced echelon form $R$ (rank, free variables, pivot variables, special solutions, etcetera)
8. Solving $Ax = 0$ for nonsquare $A$ (particular solutions, relationship of rank/nullspace/columnspace to existence and uniqueness of solutions)

If there is one central technique in all of these lectures, it is **elimination**. You should know elimination forwards and backwards. Literally: we might give you the final steps and ask you to work backwards, or ask you what properties of $A$ you can infer from certain results in elimination. Know how elimination relates to nullspaces and column spaces: elimination doesn't change the nullspace, which is why we can solve $Rx = 0$ to get the nullspace, while it does change the column space...but you can check that $b$ is in the column space of $A$ by elimination (if elimination produces a zero row from $A$, the same steps should produce a zero row from $b$ if $b$ is in the column space). Understand *why* elimination works, not just *how*. Know how/why elimination corresponds to matrix operations (elimination matrices and $L$).

One common mistake that I've warned you about before is: *never compute the inverse of a matrix*, unless you are *specifically asked to*. If you find yourself calculating $A^{-1}$ in order to compute $x = A^{-1}b$, you should instead solve $Ax = b$ for $x$ by elimination & backsubstitution. Computing the inverse matrix explicitly is a lot more work, and more error prone...and fails completely if $A$ is singular or nonsquare.

## Some practice problems

The 18.06 web site has exams from previous terms that you can download, with solutions. I've listed a few practice exam problems that I like below, but there are plenty more to choose from. Note, however that there will be *no questions asking explicitly about linear independence, basis, dimension, or the row space or left nullspace.* Reviewing the homework and solutions is always a good idea, too. The exam will consist of 3 or 4 questions (perhaps with several parts each), and you will have one hour.

1. $A$ is a $4 \times 4$ matrix with rank 2, and $Ax = b$ for some $b$ has three solutions $x = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 2 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 5 \\ 2 \\ 1 \\ 4 \end{pmatrix}$.

   Give the nullspace $N(A)$.

2. If we do a sequence of column operations (adding multiples of one column to another column) on a square matrix $A$ and obtain the identity matrix $I$, then what do we get if we do the same sequence of column operations on $A^{-1}$? (Express your answer in terms of $A$ and/or $A^{-1}$.)

3. If $A$ is $5 \times 3$, $B$ is $4 \times 5$, and $C(A) = N(B)$, then what is $BA$?

4. If $A$ and $B$ are matrices of the same size and $C(A) = C(B)$, does $C(A+B) = C(A)$? If not, give a counter-example.

5. (From spring 2007, exam 1 problem 1.) Are the following sets of vectors in $\mathbb{R}^3$ subspaces? Explain your answers.

   (a) vectors $(x,y,z)^T$ such that $2x - 2y + z = 0$
   (b) vectors $(x,y,z)^T$ such that $x^2 - y^2 + z = 0$
   (c) vectors $(x,y,z)^T$ such that $2x - 2y + z = 1$
   (d) vectors $(x,y,z)^T$ such that $x = y$ **and** $x = 2z$
   (e) vectors $(x,y,z)^T$ such that $x = y$ **or** $x = 2z$

6. (From spring 2007, exam 1 problem 3.) Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 5 \\ 1 & 3 & 5 & 9 \end{pmatrix}$.

   (a) What is the rank of $A$?
   (b) Find a matrix $B$ such that the column space $C(A)$ of $A$ equals the nullspace $N(B)$ of $B$.
   (c) Which of the following vectors belong to the column space $C(A)$? $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.

7. (From spring 2007, exam 1 problem 4.) Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & k \end{pmatrix}$.

   (a) For which values of $k$ will the system $Ax = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$ have a unique solution?
   (b) For which values of $k$ will the system from (a) have an infinite number of solutions?
   (c) For $k = 4$, find the $LU$ decomposition of $A$.
   (d) For all values of $k$, find the complete solution to $Ax = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$. (You might have to consider several cases.)

8. (From fall 2006 exam 1, problem 4.)

   (a) If $A$ is a 3-by-5 matrix, what information do you have about the nullspace of $A$?

   (b) In the vector space $M$ of all $3 \times 3$ matrices, what subspace is spanned by all possible row-reduced echelon forms $R$?

9. (From spring 2006 exam 1, problem 3.) [Hint: best if you don't work too hard on this problem!] Let

$$A = \begin{pmatrix} 1 & a & 0 & d & 0 & g \\ 0 & b & 1 & e & 0 & h \\ 0 & c & 0 & f & 1 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}.$$

   (a) Find the complete solution to $Ax = v$ if $s = 1$.

   (b) Find the complete solution to $Ax = v$ if $s = 0$.

10. (From spring 2005 exam 1, problem 1.) Suppose $A$ is reduced by the usual row operations to

$$R = \begin{pmatrix} 1 & 4 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

    Find the complete solution (if any exists) to this system involving the original $A$:

$$Ax = \text{sum of the columns of } A.$$

11. (From spring 2005 exam 1, problem 2.) Suppose the $4 \times 4$ matrices $A$ and $B$ have the *same column space*. They may not have the same columns!

    (a) Are they certain to have the same number of pivots? YES or NO. Explain.

    (b) Are they certain to have the same nullspace? YES or NO. Explain.

    (c) If $A$ is invertible, are you sure that $B$ is invertible? YES or NO. Explain.

12. (From spring 2005 exam 1, problem 3.)

    (a) Reduce $A$ to an upper-triangular matrix $U$ and carry out the same elimination steps on the right side $b$:

$$(A\, b) = \begin{pmatrix} 3 & 3 & 1 & b_1 \\ 3 & 5 & 1 & b_2 \\ -3 & 3 & 2 & b_3 \end{pmatrix} \to (U\, c).$$

    Factor the $3 \times 3$ matrix $A$ into $LU$ (lower triangular times upper triangular).

    (b) If you change the last (lower-right) entry in $A$ from 2 to _____ to get a new matrix $A_{\text{new}}$, then $A_{\text{new}}$ becomes singular. *Fill in the blank, and describe its column space exactly.*

    (c) In that singular case from (b), what conditions on $b_1$, $b_2$, and $b_3$ allow $A_{\text{new}}x = b$ to be solved?

    (d) Write down the complete solution to $A_{\text{new}}x = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$ (the first column of $A_{\text{new}}$).

## Solutions

The solutions for all problems from previous exams are posted on the 18.06 web page. Solutions to the first four problems are:

1. The differences between the solutions must be in the nullspace. We have three solutions, hence two differences: $\begin{pmatrix} 2 \\ -1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 2 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -2 \\ 0 \end{pmatrix}$. The rank of $A$ is 2 and it has 4 columns, so we only need two independent nullspace vectors to span the nullspace. Hence the nullspace is the span of these two difference vectors (which clearly aren't multiples of one another).

2. A sequence of column operations corresponds to multiplying $A$ on the right by some matrix $E$, like in the problem sets. But if $AE = I$, then $E$ must be $A^{-1}$. Doing the same operations on $A^{-1}$ gives $A^{-1}E = A^{-1}A^{-1} = A^{-2}$.

3. $BA$ is a $4 \times 3$ matrix. Since $C(A) = N(B)$, then $BAx$ for any $x$ gives $B$ multiplied by something in $N(B)$, which gives zero. Since $BAx = 0$ for any $x$, we must have $BA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

4. No. A simple example is $B = -A$ for any nonzero $A$. $C(-A) = C(A)$ (it's the same columns, just multiplied by $-1$), but $A + (-A) = 0$ and the column space of the zero matrix is just $\{0\} \neq C(A)$.