18.06	Professor Johnson	FINAL EXAM	May 19, 2009

Your PRINTED name is: ______________________________

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Grading	
1	
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Total:	

I AGREE NOT TO DISCUSS THE CONTENTS OF THIS EXAM WITH ANY STUDENTS WHO HAVE NOT YET TAKEN IT UNTIL AFTER WEDNESDAY, MAY 20.

______________________________ (YOUR SIGNATURE)

1 (18 pts.) A sequence of numbers $f_0, f_1, f_2, \ldots$ is defined by the recurrence

$$f_{k+2} = 3f_{k+1} - f_k,$$

with starting values $f_0 = 1$, $f_1 = 1$. (Thus, the first few terms in the sequence are $1, 1, 2, 5, 13, 34, 89, \ldots$.)

(a) Defining $\mathbf{u}_k = \begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix}$, re-express the above recurrence as $\mathbf{u}_{k+1} = A\mathbf{u}_k$, and give the matrix A.

(b) Find the eigenvalues of A, and use these to predict what the ratio f_{k+1}/f_k of successive terms in the sequence will approach for large k.

(c) The sequence above starts with $f_0 = f_1 = 1$, and $|f_k|$ grows rapidly with k. Keep $f_0 = 1$, but give a *different* value of f_1 that will make the sequence (with the *same recurrence* $f_{k+2} = 3f_{k+1} - f_k$) approach *zero* ($f_k \to 0$) for large k.

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2 (18 pts.) For the matrix $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ with rank 2, consider the system of equations $A\mathbf{x} = \mathbf{b}$.

(i) $A\mathbf{x} = \mathbf{b}$ has a solution whenever $\mathbf{b}$ is orthogonal to some nonzero vector $\mathbf{c}$. Explicitly compute such a vector $\mathbf{c}$. Your answer can be multiplied by any overall constant, because $\mathbf{c}$ is any basis for the ______________________ space of A.

(ii) Find the orthogonal projection $\mathbf{p}$ of the vector $\mathbf{b} = \begin{pmatrix} 9 \\ 9 \\ 9 \end{pmatrix}$ onto $C(A)$.

(*Note*: The matrix A^TA is singular, so you *cannot* use your formula $P = A(A^TA)^{-1}A^T$ to obtain the projection matrix P onto the column space of A. But I have repeatedly discouraged you from computing P explicitly, so you don't need to be reminded anyway, right?)

(iii) If $\mathbf{p}$ is your answer from (ii), then a solution $\mathbf{y}$ of $A\mathbf{y} = \mathbf{p}$ minimizes what? [You need not answer (ii) or compute $\mathbf{y}$ for this part.]

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3 (12 pts.) True or false. Give a counter-example if *false*. (You need not provide a reason if true.)

(a) If Q is an orthogonal matrix, then $\det Q = 1$.

(b) If A is a Markov matrix, then $d\mathbf{u}/dt = A\mathbf{u}$ approaches some finite constant vector (a "steady state") for any initial condition $\mathbf{u}(0)$.

(c) If S and T are subspaces of $\mathbb{R}^2$, then their intersection (points in *both* S *and* T) is also a subspace.

(d) If S and T are subspaces of $\mathbb{R}^2$, then their union (points in *either* S *or* T) is also a subspace.

(e) The rank of AB is less than or equal to the ranks of A and B for any A and B.

(f) The rank of $A + B$ is less than or equal to the ranks of A and B for any A and B.

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4 (18 pts.) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}$$

(a) Find an orthonormal basis for $C(A)$ using Gram-Schmidt, forming the columns of a matrix Q.

(b) Write each step of your Gram-Schmidt process from (a) as a multiplication of A on the __________ (*left* or *right*) by some invertible matrix. Explain how the product of these invertible matrices relates to the matrix R from the QR factorization $A = QR$ of A.

(c) Gram-Schmidt on another matrix B (of the same size as A) gives the *same* orthonormal basis (the same Q) as in part (a). Which of the four subspaces, if any, must be the same for the matrices AA^T and BB^T? [*You can do this part without doing (a) or (b).*]

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5 (16 pts.) The complete solution to $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

for any arbitrary constants c and d.

(i) If A is an $m \times n$ matrix with rank r, give as much true information as possible about the integers m, n, and r.

(ii) Construct an explicit example of a possible matrix A and a possible right-hand side $\mathbf{b}$ with the solution $\mathbf{x}$ above. (There are many acceptable answers; you only have to provide one.)

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6 (18 pts.) Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

(i) A has one eigenvalue $\lambda = -1$, and the other eigenvalue is a double root of $\det(A - \lambda I)$. What is the other eigenvalue? (Very little calculation required.)

(ii) Is A defective? Why or why not?

(iii) Using the above A, suppose we want to solve the equation

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} + c\mathbf{u}$$

where c is some real number, for some initial condition $\mathbf{u}(0)$.

(a) For what values of c will the solutions $\mathbf{u}(t)$ always to go zero as $t \to \infty$?

(b) For what values of c will the solutions $\mathbf{u}(t)$ typically diverge ($\|\mathbf{u}(t)\| \to \infty$) as $t \to \infty$?

(c) For what values of c will the solutions $\mathbf{u}(t)$ approach a constant vector (possibly zero) as $t \to \infty$?

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