18.06 Problem Set 6

Due Wednesday, 9 April 2008 at 4 pm in 2-106.

Problem 1: a) Do problem 2 from section 6.1 (pg. 283) in the book.

b) Do problem 9 from section 6.1 (pg. 284).

$\boxed{\text{Solution}}$ (5+5 points)

a) To find the eigenvalues of A, we take the determinant of $A - \lambda I$, where λ is a variable:

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) - 8$$

This simplifies to $\lambda^2 - 4\lambda - 5$, which has roots $5, -1$. (A shortcut to find this equation for a 2 by 2 matrix is to take $\lambda^2 - tr(A)\lambda + \det(A)$.) We then find the eigenvectors with eigenvalue 1 by taking the nullspace of $A - (5)I$: the nullspace of

$$A - 5I = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}$$

is given by the basis $[1, 1]^T$. Similarly,

$$A - (-1)I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$$

has nullspace given by the basis $[-2, 1]^T$. In sum, the eigenvalue 5 leads to eigenvector $(1, 1)$, and the eigenvalue -1 leads to the eigenvector $(-2, 1)$.

We now consider $A + I$. The equation giving the eigenvalues is $\lambda^2 - 6\lambda = 0$. This has roots $0, 6$. Note that $(A+I) - 0I$ and $(A+I) - 6I$ are exactly the matrices we had as before. So they have the same nullspaces. That is, $A + I$ has the same eigenvectors as A, but the eigenvalues have increased by 1.

b) Part 1: if $Ax = \lambda x$, then x is also an eigenvector of A^2 with eigenvalue λ^2, because

$$\begin{aligned} A^2 x &= A(\lambda x) \\ &= \lambda(Ax) \\ &= \lambda(\lambda x) \end{aligned}$$

Part 2: Supposing that A is invertible, x is an eigenvector of A^{-1} with eigenvalue λ^{-1}: we just take the equation $Ax = \lambda x$ and multiply both sides by $\lambda^{-1}A^{-1}$ to get $\lambda^{-1}x = \lambda^{-1}A^{-1}(\lambda x) = A^{-1}x$.

Part 3: If $Ax = \lambda x$, then $(A + I)x = \lambda x + x = (\lambda + 1)x$.

Problem 2: Do problem 13 from section 6.1 (pg. 285) in the book.

$\boxed{\text{Solution}}$ (10 points)

a) We have $Pu = uu^T u = u$ because $u^T u = u \cdot u = 1$. So u is an eigenvector with eigenvalue 1.

b) If v is perpendicular to u then $0 = u \cdot v = u^T v$. So $Pv = uu^T v = 0$, showing that v has eigenvalue 0.

c) By part b, it suffices to find three independent vectors perpendicular to u (and in fact this will be equivalent). Using the fact that the row space and null space are perpendicular, we need to find three independent vectors in the nullspace of $[1/6, 1/6, 3/6, 5/6]$. We obtain special solutions $[-1, 1, 0, 0]^T$, $[-3, 0, 1, 0]^T$, and $[-5, 0, 0, 1]^T$.

Problem 3: Consider the matrix

$$M = \begin{bmatrix} 2 & 2 & 1 & 1 \\ -14 & -6 & -9 & -7 \\ -2 & -1 & -2 & -1 \\ 8 & 1 & 7 & 4 \end{bmatrix}$$

a) One eigenvector is $x_1 = (1, 1, 0, -3)$. What is the corresponding eigenvalue?

b) Note that $\det(M) = 0$. Use this information to find another eigenvalue λ_2 - how do you know this must be an eigenvalue?

c) A third eigenvalue is $\lambda_3 = -1$. Write down (but don't solve) a linear system that can be solved to find x_3.

d) What is the fourth eigenvalue? (Hint: use the trace.)

$\boxed{\text{Solution}}$ (10 points)

a) Since x_1 is an eigenvector, we have $Mx_1 = \lambda_1 x_1$ for the corresponding eigenvalue λ_1. So we just calculate $Mx_1 = [1, 1, 0, -3]^T$. Thus $\lambda_1 = 1$.

b) We know that $\det(M)$ is the product of the eigenvalues. If $\det(M) = 0$, then one eigenvalue must be 0. (Another way of thinking about it: if M is not invertible, then it has a non-trivial nullspace, which means that it has eigenvectors with eigenvalue 0.) So we get $\lambda_2 = 0$.

c) To find the eigenvector corresponding to -1, we would need to solve the equations $(M - (-1)I)x = 0$.

d) The eigenvalues add up to the trace. The trace is the sum of the diagonal entries; in this case, $tr(M) = -2$. So our fourth eigenvalue is -2.

Problem 4: a) Do problem 8 in section 6.2 (pg. 299)

b) Do problem 18 in section 6.2 (pg. 300)

$\boxed{\text{Solution}}$ (5+5 points)

a) If a matrix has linearly independent eigenvectors, then it can be diagonalized. So every matrix with eigenvectors $(1, 1)$ and $(1, -1)$ can be diagonalized to give

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

Simplifying, we find

$$A = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{bmatrix} \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} (\lambda_1 + \lambda_2)/2 & (\lambda_1 - \lambda_2)/2 \\ (\lambda_1 - \lambda_2)/2 & (\lambda_1 + \lambda_2)/2 \end{bmatrix}$$

b) The matrix A only has eigenvalue 3. The corresponding eigenvectors are the nullspace of $A - 3I$. However, this matrix has rank 1 (in fact the only eigenvectors are $(a, 0)$). So, we can't find two linearly independent eigenvectors, and A is not diagonalizable.

To make it diagonalizable, we could change any entry but the top-right one arbitrarily (we could also change the top right entry to 0). For example, we could change the top-left 3 to a 2. The new matrix has different eigenvalues $2, 3$, and so it is automatically diagonalizable.

Problem 5: Here's an example of an invertible 3 by 3 matrix with only 2 different eigenvalues:

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$$

a) Find the eigenvalues of A.

b) Find 3 linearly independent eigenvectors of A.

c) Is A diagonalizable? If so, write down a diagonalization $A = S\Lambda S^{-1}$.

$\boxed{\text{Solution}}$ (4+4+2 points)

We find the eigenvalues of A by computing the determinant of $A - \lambda I$:

$$\begin{aligned}
\det \begin{bmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{bmatrix} &= (4-\lambda)(\lambda^2 - 7\lambda + 12) - (4 - 2\lambda + 2) + (-1)(2 - 5 + \lambda) \\
&= -\lambda^3 + 11\lambda^2 - 39\lambda + 45
\end{aligned}$$

This equation has roots $3, 3, 5$ (that is, the root 3 has multiplicity 2). So these are the eigenvalues.

Here's another way: since you know there are only two different eigenvalues, you can use the trace and determinant equations. Let λ_1 be the double eigenvalue, so that

$$\lambda_1^2 \lambda_2 = \det(A)$$

$$2\lambda_1 + \lambda_2 = tr(A)$$

Solving these two equations give the same answer.

b) To find the eigenvectors with eigenvalue 3, we find the nullspace of $A - 3I$:

$$\begin{bmatrix} 4-3 & 1 & -1 \\ 2 & 5-3 & -2 \\ 1 & 1 & 2-3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $A - 3I$ has rank 1, meaning that we can find two linearly independent vectors in the nullspace. The easiest way is just to take the basis given by the special solutions: $e_1 = [-1, 1, 0]^T$ and $e_2 = [1, 0, 1]^T$.

To find the eigenvectors with eigenvalue 5, we find the nullspace of $A - 5I$:

$$\begin{aligned}
\begin{bmatrix} 4-5 & 1 & -1 \\ 2 & 5-5 & -2 \\ 1 & 1 & 2-5 \end{bmatrix} &\rightsquigarrow \begin{bmatrix} -1 & 1 & -1 \\ 0 & 2 & -4 \\ 0 & 2 & -4 \end{bmatrix} \\
&\rightsquigarrow \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \\
&\rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

We have one special solution, $e_3 = [1, 2, 1]^T$.

c) Because we can find 3 linearly independent eigenvectors for A, it is diagonalizable. We can take Λ to be the matrix with $3, 3, 5$ on the diagonal, and S to be the matrix with columns e_1, e_2, e_3.

Problem 6: Do problems 15 and 16 in section 6.2 (pg. 300).

$\boxed{\text{Solution}}$ (5+5 points)

If the eigenvalues of A are $2, 2, 5$, then:

1. True. A is invertible, because it has no vectors with eigenvalue 0.

2. False. A may be non-diagonalizable. If it had three different eigenvalues it would be diagonalizable, but with only 2 we can't tell.

3. False. A may be diagonalizable. Again, the fact that there is a repeated eigenvalue doesn't automatically mean that A is not diagonalizable (the previous problem gave an example).

If the only eigenvectors of A are multiples of $(1, 4)$, then:

1. False. A may or may not have an inverse. You can't tell, because you don't know if it has any eigenvalues of 0 or not.

2. True. A must have a repeated eigenvalue. If it had two different eigenvalues, it would have two linearly independent eigenvectors.

3. True. A is diagonalizable if and only if A has 2 linearly independent eigenvectors, but it only has 1.

Problem 7: Do problem 22 in section 6.2 (pg. 301).

$\boxed{\text{Solution}}$ (10 points)

We find the eigenvalues of A by solving the equation $\det(A - \lambda I) = 0$. This equation is $(2 - \lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = 0$, so A has eigenvalues 3 and 1. The corresponding eigenvectors are the nullspaces of $A - 3I$ and $A - I$; they turn out to be $[1, 1]^T$ and $[1, -1]^T$ respectively. So

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Thus

$$\begin{aligned} A^k &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3^k & 0 \\ 0 & 1^k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 3^k+1 & 3^k-1 \\ 3^k-1 & 3^k+1 \end{bmatrix} \end{aligned}$$

Problem 8: Do problem 7 in section 8.3 (pg. 429).

$\boxed{\text{Solution}}$ (10 points)

Since M is a Markov matrix, it has one eigenvalue $\lambda_1 = 1$. We can find the other using the trace: $tr(A) = 1.5$ so the other eigenvalue is 0.5. Of course we could also find this directly.

The eigenvector for 1 is an element of the nullspace of

$$A - I = \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix}$$

The nullspace has basis $[0.6, 0.4]^T$ (or any multiple; usually when we work with Markov matrices we normalize our eigenvectors so the columns add up to 1). Similarly, the eigenvector for 0.5 is an element of the nullspace of $A - 0.5I$, which has basis $[0.5, -0.5]^T$.

A is diagonalized by the matrices

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$S = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & -0.5 \end{bmatrix}$$

We can now calculate A^{16}:

$$A^{16} = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & -0.5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2^{16} \end{bmatrix}\begin{bmatrix} 0.6 & 0.5 \\ 0.4 & -0.5 \end{bmatrix}^{-1}$$

Problem 9: Do problem 8 in section 8.3 (pg. 429).

$\boxed{\text{Solution}}$ (10 points)

In problem 7, we found that the steady state vector of A is $[0.6, 0.4]^T$. Now, consider A^k for very large k. The first column of this matrix is A^{k-1} acting on $[0.8, 0.2]^T$, but of course this goes to the steady state as k increases. Similarly, the second column of this matrix is A^{k-1} acting on $[0.3, 0.7]^T$, which also goes to the steady state.

The challenge problem is optional. It is very similar to problem 4a from this problem set. We need to find Markov matrices with eigenvector $[0.6, 0.4]$ corresponding to eigenvalue 1, and we can find this using the diagonalization just as before.

Problem 10: A Matlab question: The page rank algorithm in Google is essentially solving an eigenvalue problem for a matrix M with size in the billions. The method is discussed on pages 358-359 of the textbook; you can find more information in an article by Cleve Moler (MATLAB founder):

`www.mathworks.com/company/newsletters/news_notes/clevescorner/oct02_cleve.html`

The idea is to start crawling randomly from a website and count the frequency of hitting each site. We create an adjacency matrix that represents the links between websites. By rescaling the columns, we obtain a Markov matrix M - it tells us the probability of getting to a website by following a random link. If we act by M repeatedly, vectors will tend to the steady state vector. We'll call this the evector. The evector represents the total frequency of links to a site, and so sites with larger entries should have higher page ranks. Google finds the evector by crawling randomly through sites.

Model this with a 6 by 6 Markov matrix M and print the output:

```
W=ceil(rand(6) - .55*ones(6))       % create a 1-0 web link matrix W
M=W*diag(1./sum(W))   % Markov with column sums = 1  Check sum(M)
[S,L]=eig(M)          % S = eigenvector matrix of M and L = eigenvalues
x=S(:,1); v=x/sum(x)  % first column is usually evector v>0 for evalue=1
```

Start from the first website :

```
u=[1,0,0,0,0,0]'
```

Now Mu is the first column of M. Using the column Mu, figure out the probabilities of reaching site 1 to 6.

Define a vector f that is the fraction of times you hit each of the websites as you continue to crawl. I think f should approach the evector v if you act by M enough times. Does it?

$\boxed{\text{Solution}}$ (10 points)

I copied my session below. As you can see, the code yields a Markov matrix M. The matrices S and L encode the eigenvector and eigenvalue information (they are just the $M = S\Lambda S^{-1}$ matrices). Since 1 is the highest eigenvalue, it corresponds to the first columns of S and L, so the first column of S is (almost always) the steady state. The code sets v to be this column.

In my case, there are four links from website 1, leading to websites 1 (it's a narcissistic website), 3, 5, and 6. So, they each get an equal probability 0.25.

Then f represents $M^k u$ as k increases. It should approach the steady state v if k is big enough. Comparing v and $M^{50}u$, we see that they are practically equal.

```
EDU>> W = ceil(rand(6)-.55*ones(6))
W =
     1     0     1     0     0     0
     0     0     1     1     0     1
     1     1     0     0     0     0
     0     0     0     0     1     1
     1     1     1     1     0     0
     1     1     1     0     0     0
EDU>> M=W*diag(1./sum(W))
M =
    0.2500         0    0.2500         0         0         0
         0         0    0.2500    0.5000         0    0.5000
    0.2500    0.3333         0         0         0         0
         0         0         0         0    1.0000    0.5000
    0.2500    0.3333    0.2500    0.5000         0         0
    0.2500    0.3333    0.2500         0         0         0
EDU>> [S,L]=eig(M)
S =
  Columns 1 through 3
  -0.0571              0.0706 + 0.0802i   0.0706 - 0.0802i
  -0.4712              0.6103             0.6103
  -0.1713             -0.3115 - 0.1838i  -0.3115 + 0.1838i
  -0.6425             -0.2923 + 0.4745i  -0.2923 - 0.4745i
```

```
  -0.5354             0.1518 - 0.3091i   0.1518 + 0.3091i
  -0.2142            -0.2289 - 0.0618i  -0.2289 + 0.0618i
  Columns 4 through 6
  -0.4982             0.6463             0.4607
   0.1149            -0.5735             0.1455
  -0.2346            -0.1558            -0.8259
   0.5959             0.2215             0.1393
   0.4161             0.2226            -0.1356
  -0.3941            -0.3612             0.2159
L =
  Columns 1 through 3
   1.0000                  0                  0
        0            -0.5546 + 0.2629i        0
        0                  0            -0.5546 - 0.2629i
        0                  0                  0
        0                  0                  0
        0                  0                  0
  Columns 4 through 6
        0                  0                  0
        0                  0                  0
        0                  0                  0
   0.3677                  0                  0
        0             0.1897                  0
        0                  0            -0.1982
EDU>> x=S(:,1);v=x/sum(x)
v =
    0.0273
    0.2253
    0.0819
    0.3072
    0.2560
    0.1024
EDU>> u=[1,0,0,0,0,0]'
u =
     1
     0
     0
     0
     0
```

```
     0
EDU>> M*u
ans =
    0.2500
         0
    0.2500
         0
    0.2500
    0.2500
EDU>> M^50*u
ans =
    0.0273
    0.2253
    0.0819
    0.3072
    0.2560
    0.1024
```