18.06 Problem Set 5

Due Wednesday, 19 March 2008 at 4 pm in 2-106.

Problem 1: Do problem 12 from section 4.3 (pg. 217) in the book. There is a typo in part c. It should read something like this:

c) Let's analyze what happens when $b = (1, 2, 6)$. In this case $\widehat{x} = 3$ and the projection onto the line is $p = (3, 3, 3)$. Check that p is perpendicular to e. Also find the projection matrix P.

Problem 2: Do problem 17 from section 4.3 (pg. 217) in the book.

Problem 3: Find a function of the form $f(t) = C\sin(t) + D\cos(t)$ that approximates the three points $(0, 0)$, $(\pi/2, 2)$, and $(\pi, 1)$. As explained in the book, the method is the same as for fitting a line using least-squares! (See pg. 212 for a quadratic example.) The difference is that the matrix A we use will no longer have columns with entries 1 and t_i but rather $\sin(t_i)$ and $\cos(t_i)$.

Problem 4: a) Show that if Q is orthogonal (i.e. Q is square with orthonormal columns) then so is Q^T. Use the criteria $Q^TQ = I$.

b) If Q_1 and Q_2 are orthogonal, show that their product Q_1Q_2 is as well.

c) If Q has orthonormal columns but is not square, does Q^T have the same property? What can you say about QQ^T and Q^TQ in this case? Give an example for a 3 by 2 matrix Q.

Problem 5: Do problem 15 from section 4.4 (pg. 229). (Hint for part c: it may be easiest to use the decomposition $A = QR$.)

Problem 6: Do problem 18 from section 4.4 (pg. 230). Then write down the $A = QR$ decomposition for the matrix A with columns the given vectors.

Problem 7: Do problem 3 in section 5.1 (pg. 240).

Problem 8: Do problem 8 in section 5.1 (pg. 241).

Problem 9: Find the determinants of the following matrices (Use determinant = product of pivots and/or row swaps.)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 5 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 6 \\ 0 & 3 & 0 & 0 \end{bmatrix}$$

Problem 10: Here's another Matlab question. You can download the code from the "Problem Sets" section of the webpage, or just refer to the commands below. This problem concerns the condition number c which measures the sensitivity to errors of an equation $Ax = b$. (The condition number is $c = \|A\| \, \|A^{-1}\|$ - see pg. 462 for details.) Generally you lose log c decimal places to roundoff in solving Ax=b.

This problem came to me from a GPS expert who noticed that in this least-squares problem the condition number *increases* with more observations. We fit a function of the form $C + Dt + E\sin(2\pi t)$ to a set of data points that are evenly spaced over an interval from 0 to tmax. The Matlab code will output the condition number for different tmax and observation numbers.

a) What is the 3 by 3 matrix A with 3 observations that has such a big condition number on the first line of the output?

b) What condition numbers do you get if tmax is reduced to .02? Use 3, 4, 5, 10, 100 observations as in the code.

Matlab code:

```
function fitlinesine
obs=[3,4,5,10,100]; % numbers of observations to use
tmax=[1 1/2 1/4 1/8 1/16]; % maximum values of t to use
for i=1:length(tmax) % for each tmax
  tm=tmax(i); % current tmax
  disp(sprintf('For the interval [0,%f]:',tm));
```

```
  for j=1:length(obs) % for each number of observations
    m=obs(j); % current number of observations (will be number of rows)
    t=(0:m-1)'*tm/(m-1); % m linearly spaced t's from 0 to tmax
    A=[ones(m,1) t sin(2*pi*t)]; % construct matrix A
    c=cond(A); % indicates how ill conditioned system is
    disp(sprintf('%5d observations : cond=%f',m,c));
  end
  disp(' ');
end
```