18.06 Problem Set 4

Due Wednesday, 12 March 2008 at 4 pm in 2-106.

Problem 1: Do problem 2 from section 3.5 (pg. 168) in the book.

$\boxed{\text{Solution}}$ (10 points)

We can test linear independence of vectors by putting them into the columns of a matrix A and finding the pivot columns. So, we define

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

and reduce:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, the matrix has rank 3, and thus at most three of the vectors can be linearly independent (for example the first three).

Problem 2: Do problem 17 from section 3.5 (pg. 169).

$\boxed{\text{Solution}}$ (2+2+3+3 points)

a) Any vector whose components are equal can be written

$$v = \begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

That is, V is just all scalar multiples of the vector $(1, 1, 1, 1)$, which means that the vector $(1, 1, 1, 1)$ is a basis.

b) The vectors (x_1, x_2, x_3, x_4) with components that add to zero are exactly the same as vectors in the nullspace of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

We find a basis for a nullspace by taking all of the special solutions. In this case we end up with the vectors $(-1, 1, 0, 0)$, $(-1, 0, 1, 0)$ and $(-1, 0, 0, 1)$. Of course, there are many other bases, which can be found just by looking at the space.

c) We know that V is the orthogonal complement of the subspace spanned by the vectors $(1, 1, 0, 0)$ and $(1, 0, 1, 1)$. In this situation we can use the fact that the nullspace and row space are orthogonal complements. If we define the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

then V will be the nullspace of A. We find the basis for the nullspace as usual: A reduces to

$$U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

which gives us special solutions $(-1, 1, 1, 0)$ and $(-1, 1, 0, 1)$. These form a basis for V.

d) The pivot columns of U define a basis for the column space; this gives us the two vectors $(1, 0)$ and $(0, 1)$. The special solutions define a basis for the nullspace; we get $(-1, 0, 1, 0, 0)$, $(0, -1, 0, 1, 0)$, and $(-1, 0, 0, 0, 1)$.

Problem 3: Do problem 11 from section 3.6 (pg. 181).

Solution (10 points)

a) If $Ax = b$ has no solution, then the column space can not be all of $\mathbb{R}^m$ (if it were, every b would give a solution). So, we know that the dimension of $C(A)$ is less than m: $r < m$. The rank is always less than the number of rows, so $r \leq n$ as well. We can't say anything about the relative sizes of m and n. For example, we can define the matrix A_n to be n the column vector $(1, 1)$ repeated n times; these all satisfy the criteria, but sometimes $2 > n$, and sometimes $n > 2$.

b) The dimension of the left nullspace is $m - r$, and $m - r > 0$ since $m > r$.

Problem 4: Define the following matrices:

$$A = \begin{bmatrix} -1 & 1 \\ 2 & 4 \\ 3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & -1 & -1 & 7 \\ 1 & 0 & -2/7 & 2 \end{bmatrix}$$

First write down the dimensions of the four fundamental subspaces of A and B by calculating their ranks. Then find bases for the subspaces.

Solution (10 points)

We first find the rank of A by reducing:

$$\begin{bmatrix} -1 & 1 \\ 2 & 4 \\ 3 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & 1 \\ 0 & 6 \\ 0 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & 1 \\ 0 & 6 \\ 0 & 0 \end{bmatrix}$$

So A has rank 2. This means that $C(A)$ and $C(A^T)$ both have dimension 2, $N(A)$ has dimension $2 - 2 = 0$, and $N(A^T)$ has dimension $3 - 2 = 1$.

The column space has the first two columns of A (not U!) as a basis, and the row space has the non-zero rows of the reduction U as a basis. The nullspace is just the zero space, which corresponds to an "empty basis". The left nullspace is generated by one vector; we can either find it from E or by calculating the nullspace of A^T. The matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1/2 & 1 \end{bmatrix}$$

gives $EA = U$, so the vector $(2, -1/2, 1)$ gives a basis for the left nullspace.

We do the same thing for B:

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & -1 & -1 & 7 \\ 1 & 0 & -2/7 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -7 & -3 & 7 \\ 0 & -3 & -9/7 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -7 & -3 & 7 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

So A has rank 3. This means that $C(A)$ and $C(A^T)$ both have dimension 3, $N(A)$ has dimension $4-3=1$, and $N(A^T)$ has dimension $3-3=0$.

The column space has the columns 1,2, and 4 of A (not U!) as a basis, and the row space has all three rows of the reduction U as a basis. The left-nullspace is just the zero space, which corresponds to an "empty basis". The nullspace is generated by the special solution $(2/7, -3/7, 1, 0)$.

Problem 5: Do problem 3 parts a),c) from section 4.1 (pg. 191).

Solution (5+5 points)

a) We may as well pick the vectors $(1, 2, -3)$ and $(2, -3, 5)$ to be the first two columns of A. Then, if $(1, 1, 1)$ is in the nullspace, this tells us that the sum of the columns must be 0. This means that A should be

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$$

b) The question asks for a matrix A with $(1, 1, 1)$ in the column space and $(1, 0, 0)$ in the left nullspace. However, these two vectors are not perpendicular, which means that no such A can exist. A more concrete way of thinking about it: if $(1, 0, 0)$ is in the left nullspace, this means that the top row of A is all 0. But then $(1, 1, 1)$ can't be in the column space; every column has a 0 in the top spot.

Problem 6: Do problem 21 from section 4.1 (pg. 193).

Solution (10 points)

We find orthogonal complements by putting the vectors into the rows of a matrix A and calculating the nullspace. Thus, this problem is equivalent to finding the nullspace of

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$$

We reduce to get

$$U = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

which yields special solutions $(0, -1, 1, 0)$ and $(-5, 1, 0, 1)$.

Problem 7: a) Project the vector $(2, 7, 3)$ onto the line going through the origin and $(1, 1, 1)$.

b) Project the vector $(2, 4, 5)$ onto the column space of the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$\boxed{\text{Solution}}$ (5+5 points)

a) Define the vector

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Then, the projection of the vector

$$b = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}$$

is given by the equation $Pb = a(a^T a)^{-1} a^T b$. We can rewrite this using the dot product:

$$Pb = \frac{a \cdot b}{a \cdot a}\, a$$

We have $a \cdot b = 12$ and $a \cdot a = 3$, giving us a projection of $4a = (4, 4, 4)$.

b) If we just have one projection to do, it is often a little easier computationally to use $\widehat{x}$ instead of calculating P (of course either method works fine). Define

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

We know that $A^TA\widehat{x} = A^Tb$. Calculating:

$$\begin{aligned} A^TA &= \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \\ \widehat{x} &= (A^TA)^{-1}A^Tb \\ &= \frac{1}{2}\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 5 \end{bmatrix} \end{aligned}$$

This means that the projection of b is given by adding -2 times the first column of A to 5 times the second column of A:

$$Pb = A\widehat{x} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix}$$

Problem 8: a) Do problem 13 in section 4.2 (pg. 204).
b) Do problem 27 in section 4.2 (pg. 205).

Solution (5+5 points)

a) Projecting onto the column space of A means that we are sending the fourth component of the vector to 0 (but not changing the other three). We can check:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so that $A^TA = I$. P should be a 4 by 4 matrix. In fact $P = A(A^TA)^{-1}A^T$ is the 4 by 4 identity with the bottom right 1 changed to a 0. The projection of b is $(1, 2, 3, 0)$.

b) Suppose that $A^TAx = 0$. Then the vector Ax is in the nullspace of A^T. Ax is always in the column space of A. Since these are orthogonal, Ax must be 0.

Problem 9: Do problem 8 in section 8.2 (pg. 421). (The graph is the square one at the bottom of page 420.)

Solution (10 points)

The incidence matrix has one column for each node, and one row for each edge. For each edge we put a -1 in the position of the starting node, and a 1 in the position of the ending node. We get the incidence matrix

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

We are also asked to find one vector in the nullspace and two in the left nullspace. Any incidence matrix has a nullspace with a basis given by the vector of all 1s. So, the vector $(1, 1, 1, 1)$ is in the nullspace.

To find the left nullspace, we travel around small loops and keep track of which edges we cross. Starting from point 1 on the top-left small loop and going clockwise, we go over edge 1 forwards, edge 3 forwards, and edge 2 backwards, giving us the vector $(1, -1, 1, 0, 0)$. The other loop gives $(0, 0, -1, 1, -1)$. These vectors will give a basis for the left nullspace.

Problem 10: Do problem 11 in section 8.2 (pg. 421). Use the A you just calculated for problem 8 in section 8.2.

Solution (5+5 points) We have

$$A^T A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

a) The ith diagonal entry represents how many edges enter or leave from node i.

b) The i, j off-diagonal is a -1 if there is an edge connecting node i and node j (in either direction), and 0 otherwise.