18.06 Problem Set 9 - Solutions

Due Wednesday, May 2, 2007 at **4:00 p.m.** in 2-106

Problem 1 *Wednesday 4/25*

Do problem 7 of section 8.1 in your book.

Solution 1

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} : 5 \times 4 \text{ matrix}$$

$$C = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & c_4 & 0 \\ 0 & 0 & 0 & 0 & c_5 \end{bmatrix} : 5 \times 5 \text{ matrix}$$

$$K = A^T C A = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 & 0 \\ -c_2 & c_2 + c_3 & -c_3 & 0 \\ 0 & -c_3 & c_3 + c_4 & -c_4 \\ 0 & 0 & -c_4 & c_4 + c_5 \end{bmatrix} : 4 \times 4 \text{ matrix}$$

When $C = I$, $K = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$, $Ku = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$,

then $u = K^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. $K^{-1} = \frac{1}{5} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$, so $u = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}$.

Problem 2 *Wednesday 4/25*

(a) Show that for n masses joined by $(n+1)$ springs with both ends fixed and with all spring constants $c_i = 1$, the stiffness matrix is

$$K_n = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & \cdots \\ & & \ddots & & \end{bmatrix}$$

(the $n \times n$ tridiagonal matrix with 2's on the main diagonal and -1's on the subdiagonal and the superdiagonal.)
(b) Calculate the determinant $det(K_n)$. (Hint: Try to express $det(K_n)$ in terms of $det(K_{n-1})$ and $det(K_{n-2})$.
(c) Calculate, the inverse matrix K_n^{-1}, for $n = 3, 4, 5$ and try to guess/calculate the answer for general n.
(d) Find the displacements of the n bodies. That is, solve $K_n u = [1, ..., 1]^T$

Solution 2

(a) A is $(n+1) \times n$ matrix such that $A_{i,i} = 1$, $A_{i+1,i} = -1$, 0 everywhere else.
Therefore, $K_n = A^T I A$ is a matrix such that $[K_n]_{i,i} = 2$, $[K_n]_{i,i+1} = -1$, $[K_n]_{i+1,i} = -1$, 0 everywhere else.

(b) $detK_n = 2 * detK_{n-1} - (-1) * (-1) * detK_{n-2}$ due to cofactor formula along the first row.

Since $K_1 = 2$, $K_2 = 3$, we can prove that $detK_n = n + 1$ by mathematical induction.
If $n = 1, 2$, then $detK_n = n + 1$
If $detK_n = n + 1$ holds for $n \leq m$, then $K_{m+1} = 2 * detK_m - (-1) * (-1) * detK_{m-1} = m + 2$.
By mathematical induction, $det(K_n) = n + 1$ for all n.

(c) $K_3^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$, $K_4^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$, $K_5^{-1} = \frac{1}{6}\begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$
Define R_n as follows.
For $i \geq j$, $[R_n]_{i,j} = \frac{1}{n+1} * (n + 1 - i) * j$.

For $j \geq i$, $[R_n]_{i,j} = \frac{1}{n+1} * (n + 1 - i) * j$ because K_n is symmetric.

This is the inverse of K_n because

$i \neq 1, n$
$[K_nR_n]_{i,j} = \Sigma_{r=1}^{r=n}[K_n]_{i,r} * [R_n]_{r,j} = \frac{1}{n+1} * (2 * [R_n]_{i,j} - [R_n]_{i-1,j} - [R_n]_{i+1,j})$
$= 1$ if $j = i$, 0 otherwise.

$i = 1$
$[K_nR_n]_{1,j} = \Sigma_{r=1}^{r=n}[K_n]_{1,r} * [R_n]_{r,j} = \frac{1}{n+1} * (2 * [R_n]_{1,j} - [R_n]_{2,j})$
$= 1$ if $j = 1$, 0 otherwise.

$i = n$
$[K_nR_n]_{n,j} = \Sigma_{r=1}^{r=n}[K_n]_{n,r} * [R_n]_{r,j} = \frac{1}{n+1} * ((-1) * [R_n]_{n-1,j} + 2 * [R_n]_{n,j})$
$= 1$ if $j = 1$, 0 otherwise.
(d) $u = R_n[1, ..., 1]^T$,
so $u_k = \Sigma$ elements in kth row in $R_n = \frac{\Sigma_{i=1}^{i=k}(n+1-k)*i}{n+1} + \frac{\Sigma_{i=1}^{i=n-k}k*i}{n+1} = \frac{k(n+1-k)}{2}$

Problem 3 *Friday 4/27*

Do problem 3 of section 6.6 in your book.

Solution 3

If $B = M^{-1}AM$ then $MB = AM$. Let $M = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

$\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & x+y \\ 0 & z+w \end{bmatrix} = \begin{bmatrix} x & y \\ x & y \end{bmatrix}$
This holds for $x = 0$, $y = 1$, $z = 1$, $w = 0$, $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} \leftrightarrow \begin{bmatrix} x+y & x+y \\ z+w & z+w \end{bmatrix} = \begin{bmatrix} x-z & y-w \\ -x+z & -y+w \end{bmatrix}$
This holds for $x = 0$, $y = 1$, $z = -1$, $w = 0$, $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} \leftrightarrow \begin{bmatrix} 4x+2y & 3x+y \\ 4+2w & 3z+1w \end{bmatrix} = \begin{bmatrix} x+2z & y+2w \\ 3x+4z & 3y+4w \end{bmatrix}$
This holds for $x = 3$, $y = 1$, $z = -2$, $w = 2$, $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Problem 4 *Friday 4/27*

Do problem 11 of section 6.6 in your book.

Solution 4

$dw/dt = 5w + x$, then we can guess $w = (w(0) + tx(0) + \frac{t^2}{2}y(0) + \frac{t^3}{6}z(0))e^{5t}$.
This satisfies $dw/dt = 5w + x$ because
$dw/dt = 5 * (w(0) + tx(0) + \frac{t^2}{2}y(0) + \frac{t^3}{6}z(0))e^{5t} + (x(0) + ty(0) + \frac{t^2}{2}z(0))e^{5t} = 5w + x$.

Problem 5 *Friday 4/27*

Do problem 12 of section 6.6 in your book.

Solution 5

Let $M = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$.

JM has 1st row=2nd row of M, 2nd row=0, 3rd row=4th row of M, 4rd row=0.
MK has first column=0, 2nd column=1st column of M, 3rd column=3rd column of M, 4th column=0.

$MK = \begin{bmatrix} 0 & a & b & 0 \\ 0 & e & f & 0 \\ 0 & i & j & 0 \\ 0 & m & n & 0 \end{bmatrix} = JM = \begin{bmatrix} e & f & g & h \\ 0 & 0 & 0 & 0 \\ m & n & o & p \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
Comparing each side, we can figure out $e = f = h = 0$, $m = n = p = 0$.
$\therefore$ The 2nd row and 4rd row of M is linearly dependent.
$\therefore$ M is not invertible.

Problem 6 *Friday 4/27*

Do problem 20 of section 6.6 in your book.

Solution 6

(a) Let $A = MBM^{-1}$. Then, $A^2 = MBM^{-1}MBM^{-1} = MB^2M^{-1}$, so A^2 is similar to B^2.
(b) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
Then, $A^2 = B^2 = 0$, so A^2 and B^2 are similar, but A and B have diffrent rank, so A and B are not similar.
(c)Both matrices are diagonalizable to $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, so they are similar.
(d) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is diagonalizable, but $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable, so they cannot be similar matrices.
(e)The new matrix is PAP^{-1} where P is permutation matrix P_{12}.

Problem 7 *Monday 4/30*

Do problem 7 of section 6.7 in your book.

Solution 7

$AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

eigenvalues:1,3 with eigenvectors : $w_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$, $w_3 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

eigenvalues:0,1,3 with eigenvectors $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$, $v_3 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$, $v_0 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$

$Av_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$Av_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = 1 * u_1$, where $u_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$Av_3 = \begin{bmatrix} 3/\sqrt{6} \\ 3/\sqrt{6} \end{bmatrix} = \sqrt{3} * u_3$, where $u_3 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ $\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix}$ $V^T = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$

$U\Sigma V^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Problem 8 *Monday 4/30*

Do problem 9 of section 6.7 in your book.

Solution 8

A is rank 1, 3×4 matrix. $Av = \sigma u$, so $\sigma = 12$.
Because $C(A)$ has dimension 1, we have only one singular value.
u is in the column space, so we can write $A = \begin{bmatrix} 2a & 2b & 2c & 2d \\ 2a & 2b & 2c & 2d \\ a & b & c & d \end{bmatrix}$ for some a,b,c,d.

$A^TAv = \sigma^2 v = \sigma A^T u \rightarrow \sigma v = xdvA^T u = \begin{bmatrix} 3a \\ 3b \\ 3c \\ 3d \end{bmatrix} \rightarrow a = b = c = d = 2.$

$\therefore A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 2 \end{bmatrix}$

Problem 9 *Monday 4/30*

Do problem 10 of section 6.7 in your book.

Solution 9

Suppose A is a $m \times n$ matrix. Notice that since the columns are orthogonal, they are linearly independent, so the rank of A is n and $n \leq m$.
$A^T A = diag(\sigma_1^2, \sigma_2^2, ..., \sigma_n^2)$
and eigenvector matrix for diagonal matrix $diag(\sigma_1^2, \sigma_2^2, ..., \sigma_n^2)$ is $I_{n \times n} = V$.
$\Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_n)$ augmented by $(m-n) \times n$ 0 matrix in the bottom.
U has columns such that $\sigma_i u_i = Av_i = w_i \rightarrow u_i = \sigma_i^{-1} w_i$. $(\sigma_i \neq 0)$
U is a matrix whose ith column is $\frac{1}{\sigma_i} w_i$ for $i \leq n$. For $i > n$, choose and orthonormal basis for the nullspace of A.

Problem 10 *Monday 4/30*

Do problem 15 of section 6.7 in your book.

Solution 10

(a)If $A = U\Sigma V^T$, then $4A = U * (4\Sigma) * V^T$. Here, SVD changes $(U, \Sigma, V) \rightarrow (U, 4\Sigma, V)$.
(b)If $A = U\Sigma V^T$, then $A^T = V * \Sigma * U^T$. Here, SVD changes $(U, \Sigma, V) \rightarrow (V, \Sigma, U)$.
If $A = U\Sigma V^T$, then $A^{-1} = (V^T)^{-1} * \Sigma^{-1} * U^{-1} = V\Sigma^{-1} * U^T$.
Here, SVD changes $(U, \Sigma, V) \rightarrow (V, \Sigma^{-1}, U)$.