18.06 Problem Set 7 - Solutions

Due Wednesday, April 18, 2007 at **4:00 p.m.** in 2-106

Problem 1 *Wednesday 4/11*

Do problem 9 of section 8.3 in your book.

Solution 1

We get that:
$\mathbf{u_1} = P\mathbf{u_0} = (0,0,1,0)$,
$\mathbf{u_2} = P\mathbf{u_1} = (0,1,0,0)$,
$\mathbf{u_3} = P\mathbf{u_2} = (1,0,0,0)$,
$\mathbf{u_4} = P\mathbf{u_3} = (0,0,0,1)$.
From here it is easily seen that $\mathbf{u_{4k+1}} = (0,0,1,0)$, $\mathbf{u_{4k+2}} = (0,1,0,0)$, $\mathbf{u_{4k+3}} = (1,0,0,0)$, $\mathbf{u_{4k}} = (0,0,0,1)$ for all nonnegative integers k. The four eigenvalues of P which solve $\lambda^4 = 1$ are $1, -1, i, -i$.
The steady state is not $\mathbf{u}_\infty$ because all the eigenvalues are of unit length. Thus there is no A^∞.

Problem 2 *Wednesday 4/11*

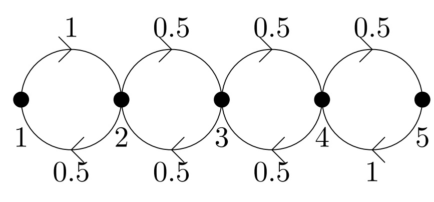
Do problem 12 of section 8.3 in your book.

Solution 2

The columns of A must sum to 1, so $A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ .2 & .3 & .5 \end{bmatrix}$.
Our theory tells us the steady state is the eigenvector with $\lambda = 1$, and sure enough there is one: $x_1 = (1,1,1)$ (or any multiple of x_1) works.
Why is $x_1 = (1,1,\dots,1)$ a steady state? The entries of Ax_1 are the sums of each row. *But A is symmetric,* so these are the same as the sums of each column, which are 1. So the entries of Ax_1 are 1, just like the entries of x_1.

Problem 3 *Wednesday 4/11*

Consider the random walk on the directed graph shown below. More precisely, there are 5 nodes and $Prob(i,i+1) = Prob(i,i-1) = 1/2$ for $i = 2,3,4$; and $Prob(1,2) = Prob(5,4) = 1$. Here $Prob(i,j)$ is the probability to go from the i-th node to the j-th node.



Find the eigenvalues and the steady state distribution for this Markov chain.

Solution 3

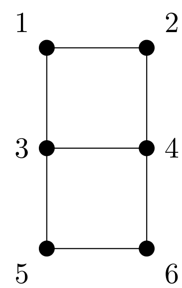
The Markov matrix corresponding to the Markov chain is

$$P = \begin{bmatrix} 0 & 05 & 0 & 0 & 0 \\ 1 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 1 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}$$

The eigenvalues of the matrix are: $1, 0.7071, 0, -1, -0.7071$. The steady state distribution is the eigenvector corresponding to the eigenvalue 1, and it is $(\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8})$.

Problem 4 *Wednesday 4/11*

Consider the 3×2 grid shown below. Assume an ant starts in vertex 1. At every step, if the ant is in vertex i, it either stays where it is with probability $\frac{1}{2i}$ or moves to an adjacent vertex selected uniformly among the current neighbors.



(a) What matrix A represents this Markov Chain?
(b) What is the sum of the eigenvalues of A?
(c) Use `MATLAB` to compute the eigenvalues of A.
(d) What is the steady state? What is the probability that in the steady state the ant is on vertex 6?

Solution 4

(a) The Markov matrix corresponding to the Markov chain is

$$A = \begin{bmatrix} \frac{1}{2} & \frac{3}{8} & \frac{5}{18} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{7}{24} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{6} & \frac{7}{24} & \frac{9}{20} & 0 \\ 0 & \frac{3}{8} & \frac{5}{18} & \frac{1}{8} & 0 & \frac{11}{24} \\ 0 & 0 & \frac{5}{18} & 0 & \frac{1}{10} & \frac{11}{24} \\ 0 & 0 & 0 & \frac{7}{24} & \frac{9}{20} & \frac{1}{12} \end{bmatrix}.$$

(b) The sum of the eigenvalues of A is equal to the trace of A so it is $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12}$.
(c) `MATLAB` says that the eigenvalues of A are $-0.6775, 1.0000, 0.6394, 0.3462, -0.1384, 0.0552$.
(d) The steady state is the eigenvector corresponding to eigenvalue 1. Using `MATLAB` we get the following:
`[V,D]=eig(A)`

$$V = \begin{bmatrix} 0.1804 & -0.5274 & 0.7246 & 0.4310 & -0.3967 & 0.3627 \\ -0.2156 & -0.3516 & 0.3405 & -0.4988 & 0.5928 & 0.0958 \\ -0.4737 & -0.4747 & -0.0959 & 0.4347 & 0.1112 & -0.7101 \\ 0.5310 & -0.4520 & -0.1664 & -0.5340 & -0.4494 & -0.3748 \\ 0.4441 & -0.2930 & -0.3954 & 0.2801 & 0.4362 & 0.2174 \\ -0.4663 & -0.2877 & -0.4073 & -0.1130 & -0.2943 & 0.4091 \end{bmatrix}$$

$$D = \begin{bmatrix} -0.6775 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6394 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3462 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.1384 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0552 \end{bmatrix}$$

Thus, the steady state is a multiple of the second column of V. For one, we must multiply by -1 first, since we must have all entries nonnegative to deal with probabilities, and then we must divide by the sum of the entries to have the sum be one. Once we do this, we get that the steady state $(.2210, .1473, .1989, .1894, .1227, .1205)$ and so the probability that in the steady state the ant is on vertex 6 is .1205. Notice that this is the lowest out of all vertices.

Problem 5 *Friday 4/13*

Do problem 2 of section 8.5 in your book.

Solution 5

Let $f(x) = 1,\ g(x) = x$ and $h(x) = x^2 - \frac{1}{3}$. In order to show that the previous functions are mutually orthogonal, we need to show that their inner products are all 0.
$(f, g) = \int_{-1}^{1} x dx = 0$
$(f, h) = \int_{-1}^{1}(x^2 - \frac{1}{3})dx = \int_{-1}^{1} x^2 dx - \int_{-1}^{1} \frac{1}{3} dx = \frac{2}{3} - \frac{2}{3} = 0$
$(g, h) = \int_{-1}^{1}(x^3 - \frac{1}{3}x)dx = \int_{-1}^{1} x^3 dx - \int_{-1}^{1} \frac{1}{3} x dx = \frac{x^4}{4}|_{-1}^{1} - \frac{1}{3}\frac{x^2}{2}|_{-1}^{1} = 0.$
We can write $2x^2 = 2(x^2 - \frac{1}{3}) + \frac{2}{3}1 = 2h(x) + \frac{2}{3}f(x)$.

Problem 6 *Friday 4/13*

Do problem 12 of section 8.5 in your book. You don't have to write the "differentiation matrix" (this involves concepts of chapter 7 that you haven't learned yet).

Solution 6

$f_1(x) = 1,\ f_2(x) = \cos x,\ f_3(x) = \sin x,\ f_4(x) = \cos 2x,\ f_5(x) = \sin 2x.$
The derivatives are:
$f_1^{'}(x) = 0$
$f_2^{'}(x) = -\sin x = -f_3(x)$
$f_3^{'}(x) = \cos x = f_2(x)$
$f_4^{'}(x) = -2\sin 2x = -2f_5(x)$
$f_5^{'}(x) = 2\cos 2x = 2f_4(x)$

Problem 7 *Friday 4/13*

Do problem 1 of section 10.2 in your book.

Solution 7

You can still find lengths by the Pythagorean theorem (since $|a + bi| = \sqrt{a^2 + b^2}$):
$\|u\| = \sqrt{(1 + 1) + (1 + 1) + (1 + 4)} = \sqrt{9} = 3,$
and $\|v\| = \sqrt{(0 + 1) + (0 + 1) + (0 + 1)} = \sqrt{3}.$
Or take the dot product (don't forget to conjugate!):
$\|u\| = \sqrt{u^{\mathsf{H}} u} = \sqrt{(1 - i)(1 = i) + (1 + i)(1 - i) + (1 - 2i)(1 + 2i)} = \sqrt{2 + 2 + 5} = 3,$
and $\|v\| = \sqrt{v^{\mathsf{H}} v} = \sqrt{(-i)(+i) + (-i)(+i) + (-i)(+i)} = \sqrt{1 + 1 + 1} = \sqrt{3}.$
For complex inner products, *order matters:*
$u^{\mathsf{H}} v = (1 - i)i + (1 + i)i + (1 - 2i)i = 2 + 3i,$
but $v^{\mathsf{H}} u = -i(1 + i) - i(1 - i) - i(1 + 2i) = 2 - 3i!$

(The difference is that $(u^H v)^H = vu^H$ conjugates u, but $v^H u$ conjugates v. So the two products are conjugates of each other.)

Problem 8 *Friday 4/13*

Do problem 10 of section 10.2 in your book.

Solution 8

The eigenvalues of P are the solutions of $\lambda^3 = 1$ which are the 3 roots of unity: $\alpha = e^{2\pi\imath/3}, \alpha^2 = e^{4\pi\imath/3}, \alpha^3 = 1$. Three corresponding eigenvectors are $(1, \alpha^2, \alpha)$, $(1, \alpha, \alpha^2)$ and $(1, 1, 1)$. The corresponding unit eigenvectors are then $\frac{1}{\sqrt{3}}(1, \alpha^2, \alpha)$, $\frac{1}{\sqrt{3}}(1, \alpha, \alpha^2)$ and $\frac{1}{\sqrt{3}}(1, 1, 1)$, since the lengths of all three vectors from the previous sentence were 3. If we put these unit vectors as columns into a matrix F we get a unitary matrix. It is an easy calculation to check that in fact the three unit eigenvectors are orthogonal. The property of P that insures this is that P is itself unitary, since being a permutation matrix $P^H P = P^T P = I$. To see that a unitary matrix has orthogonal eigenvectors, observe that every eigenvector of P is also an eigenvector of P^{-1} which in the unitary case is $P^{-1} = P^H$. We can now use the same method as for Hermitian matrices to show that the eigenvectors are orthogonal.

Problem 9 *Friday 4/13*

Do problem 16 of section 10.2 in your book.

Solution 9

To find the eigenvalues, solve $\det(K - \lambda I) = \lambda^2 - i\lambda + 2 = 0$, which yields two eigenvalues $\lambda_1 = -i$ and $\lambda_2 = 2i$, so both eigenvalues are imaginary. An eigenvector for λ_1 is $(1-i, i)$ and an eigenvector for λ_2 is $(1-i, -2i)$. The length of the first vector is $l_1 = \sqrt{3}$ and of the second is $l_2 = \sqrt{6}$. Having this in mind, $U = \begin{bmatrix} \frac{1-i}{\sqrt{3}} & \frac{1-i}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & \frac{-2i}{\sqrt{6}} \end{bmatrix}$ and $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

Problem 10 *Friday 4/13*

Do problem 17 of section 10.2 in your book.

Solution 10

First find the eigenvalues: $\lambda^2 - 2(\cos\theta)\lambda + 1 = 0$ has roots $\lambda = \cos\theta \pm i\sin\theta$. *Notice that both eigenvalues have* $|\lambda| = 1$, since Q is orthogonal.

Now find the eigenvectors. For $\lambda_+ = \cos\theta + i\sin\theta$, we want a vector x with $\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} x = 0$, such as $x_+ = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. Similarly, $\lambda_- = \cos\theta - i\sin\theta$ has eigenvector $x_- = \begin{bmatrix} 1 \\ +i \end{bmatrix}$.

These eigenvectors are automatically orthogonal (that is, $(u_+, u_-) = 1(1) - i(-i) = 1 - 1 = 0$), but we want the columns of U to be ortho*normal*, so we need to divide by the lengths: $u_+ = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $u_- = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ +i \end{bmatrix}$.

Then our factorization is

$$\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{Q} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & i/\sqrt{2} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{bmatrix}}_{U^H}$$