# 18.06 Problem Set 6 - Solutions

Due Wednesday, April 11, 2007 at **4:00 p.m.** in 2-106

**Problem 1** *Wednesday 4/4*

Do problem 9 of section 6.1 in your book.

***Solution 1***

( a) Multiply $A$ on the left to both sides of the equation $Ax = \lambda x$ to get $AAx = A\lambda x$. But $AAx = A^2x$ and $A\lambda x = \lambda Ax = \lambda\lambda x = \lambda^2 x$, so we have $A^2x = \lambda^2 x$, which means that $\lambda^2$ is an eigenvalue of $A^2$.
(b)Multiply $\lambda^{-1}A^{-1}$ on the left to both sides of the equation $Ax = \lambda x$ to get $\lambda^{-1}A^{-1}Ax = \lambda^{-1}A^{-1}\lambda x$. But $\lambda^{-1}A^{-1}Ax = \lambda^{-1}x$ and $\lambda^{-1}A^{-1}\lambda x = A^{-1}\lambda^{-1}\lambda x = A^{-1}x$, so we have $A^{-1}x = \lambda^{-1}x$, which means that $\lambda^{-1}$ is an eigenvalue of $A^{-1}$.
(c)Add $x$ to both sides of the equation $Ax = \lambda x$ to get $Ax + x = \lambda x + x$. But this is exactly $(A + I)x = (\lambda + 1)x$, which means that $\lambda + 1$ is an eigenvalue of $A + I$.

**Problem 2** *Wednesday 4/4*

Do problem 28 of section 6.1 in your book.

***Solution 2***

T he matrix $A$ has rank 1 (all rows are equal), which implies that 0 is an eigenvalue of $A$ (the three independent vectors in the nullspace of $A$ are the three independent eigenvectors with eigenvalue 0). Now let us find other eigenvalues. If $(x, y, z, w)^T$ is an eigenvector with eigenvalue $\lambda \neq 0$, then:

$$A \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x+y+z+w \\ x+y+z+w \\ x+y+z+w \\ x+y+z+w \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

But this implies that $x = y = z = w$ and furthermore $\lambda = 4$. Thus, the four eigenvalues of $A$ are $0, 0, 0, 4$.
The matrix $B$ has rank 2 (rows 1 and 3 are equal, rows 2 and 4 are equal), which implies that 0 is an eigenvalue of $A$ (the two independent vectors in the nullspace of $A$ are the two independent eigenvectors with eigenvalue 0). Now let us find other eigenvalues. If $(x, y, z, w)^T$ is an eigenvector with eigenvalue $\lambda \neq 0$, then:

$$A \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x+z \\ y+w \\ x+z \\ y+w \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

But this implies that $x = z$ and $y = w$, and furthermore $\lambda = 2$ (we get two independent eigenvectors here: $(1, 0, 1, 0)^T$ and $(0, 1, 0, 1)^T$). Thus, the four eigenvalues of $A$ are $0, 0, 2, 2$.

**Problem 3** *Wednesday 4/4*

Do problem 33 of section 6.1 in your book.

***Solution 3***

( a) Since $u, v, w$ are independent, any vector $x$ can be written as a linear combination of those, $x = c_1u + c_2v + c_3w$. Then

$$Ax = A(c_1u + c_2v + c_3w) = c_1Au + c_2Av + c_3Aw = 3c_2v + 5c_3w$$

If $Ax = 0$, then we must have $c_2, c_3 = 0$, so the vectors in the nullspace of $A$ are multiples of $u$, and a basis for $N(A)$ is the vector $u$.
All vectors $Ax$ in the column space of $A$ are linear combinations of $v$ and $w$: a basis for $C(A)$ consists of the vectors $v$ and $w$.
(b) We want to find the solutions of $Ax = v + w$. Let $x = c_1u + c_2v + c_3w$. Then as seen above $Ax = 3c_2v + 5c_3w$, so we must have $c_2 = \frac{1}{3}$ and $c_3 = \frac{1}{5}$, while $c_1$ can take any values. The solution for this is of the form $x = c_1u + \frac{1}{3}v + \frac{1}{5}w$.
(c) $Ax = u$ has no solution because if it did then $u$ would be in the column space.

**Problem 4** *Wednesday 4/4*

Let $A$ be a fixed $n \times n$ matrix. We would like to find a matrix $B$ such that $AB = BA$. This is the same as solving $AB - BA =$ zero matrix. It turns out that this is a system of $n^2$ equations on the entries of $B$ (which are unknown). Since all these equations are linear, we can associate this system to a matrix $M$. Find an eigenvector of this matrix $M$ with its corresponding eigenvalue.

***Solution 4***

We have $Mx = 0$ exactly when the vector $x$ corresponds to a matrix $B$ that satisfies $AB - BA = 0$. But there is one case of such a matrix that is quite simple: just take $B$ to be the matrix $A$ itself! Then clearly $AA - AA = 0$! So if $x$ is the vector corresponding to the matrix $A$, then $Mx = 0$, and this means that $x$ is an eigenvector of $M$, with eigenvalue 0.

**Problem 5** *Monday 4/9*

Do problem 7 of section 6.2 in your book.

***Solution 5***

W e begin by computing the eigenvalues of $A$, solving $\det(A - \lambda I) = 0$ for $\lambda$.

$$\det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix} = (4 - \lambda)(2 - \lambda)$$

The eigenvalues are $\lambda = 2$ and $\lambda = 4$.
Now, for each eigenvalue $\lambda$, we want to find the eigenvectors, i.e., vectors in the nullspace of $A - \lambda I$. For $\lambda = 2$, we have $A - 2I = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$, so $N(A - 2I)$ is generated by the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus, any vector of the form $\begin{bmatrix} 0 \\ a \end{bmatrix}$ with $a \neq 0$ is a suitable eigenvector. For $\lambda = 4$, we have $A - 4I = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$, so $N(A - 4I)$ is generated by the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Thus, any vector of the form $\begin{bmatrix} 2b \\ b \end{bmatrix}$ with $b \neq 0$ is a suitable eigenvector. Writing in these vectors as columns of a matrix we get a matrix $S$ that diagonalizes $A$:

$$S = \begin{bmatrix} 0 & 2b \\ a & b \end{bmatrix} \quad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

If we switch the columns, we still get a matrix that diagonalizes $A$:

$$S = \begin{bmatrix} 2b & 0 \\ b & a \end{bmatrix} \quad \Lambda = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

We know that if $x$ is an eigenvector of $A$ (with eigenvalue $\lambda$), then it is also an eigenvector of $A^{-1}$ (with eigenvalue $\lambda^{-1}$), so the same matrices $S$ work for diagonalizing $A^{-1}$ (the diagonal matrix changes accordingly).

**Problem 6** *Monday 4/9*

Do problem 10 of section 6.2 in your book.

***Solution 6***

T he equations $G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k$ and $G_{k+1} = G_{k+1}$ can be written in matrix form as

$$\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$$

(a)Firstly, we find the eigenvalues of $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$ by solving $\det(A - \lambda I) = 0$ for $\lambda$:

$$\det(A - \lambda I) = \det \begin{bmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ 1 & -\lambda \end{bmatrix} = (\lambda - 1)(\lambda + \frac{1}{2})$$

The eigenvalues are $\lambda = 1$ and $\lambda = -\frac{1}{2}$.
Now, we find the eigenvectors for each $\lambda$. For $\lambda = 1$, we have $A - I = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{bmatrix}$, so $N(A - I)$ is generated by the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and this is an eigenvector. For $\lambda = -\frac{1}{2}$, we have $A + \frac{1}{2}I = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$, so $N(A + \frac{1}{2}I)$ is generated by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and this is another eigenvector.
(b) The eigenvector matrix is $S = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$, its inverse is $S^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$, and the eigenvalue matrix is $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$. Then $A^n = S\Lambda^n S^{-1}$. As $n \to \infty$,

$$\Lambda^n = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}^n = \begin{bmatrix} 1^n & 0 \\ 0 & (-\frac{1}{2})^n \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Then,

$$A^n = S\Lambda^n S^{-1} \to \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

(c) Applying $A$ repeatedly to $\begin{bmatrix} G_1 \\ G_0 \end{bmatrix}$ we get

$$\begin{bmatrix} G_{n+1} \\ G_n \end{bmatrix} = A^n \begin{bmatrix} G_1 \\ G_0 \end{bmatrix}$$

But $A^n \begin{bmatrix} G_1 \\ G_0 \end{bmatrix} \to \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$, which implies that $\begin{bmatrix} G_{n+1} \\ G_n \end{bmatrix} \to \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$, that is, the Gibonacci numbers $G_n$ approach $\frac{2}{3}$.

**Problem 7** *Monday 4/9*

Do problems 15 and 16 of section 6.2 in your book.

***Solution 7***

Problem 15
If the eigenvalues of $A$ are $2, 2, 5$ then the matrix is certainly invertible, as its determinant is $\det A = 2 \times 2 \times 5 = 20 \neq 0$. Such a matrix could be diagonalizable or not, depending on whether or not there are two independent eigenvectors for the eiegnvalue 2.

Problem 16
If the only eigenvectors of $A$ are multiples of $(1, 4)$, i.e., there is only one independent eigenvector, then $A$ must have a repeated eigenvalue, as eigenvectors corresponding to distinct eigenvalues are independent. This matrix is not diagonalizable, since there aren't enough independent eigenvectors (we needed two of them for this 2-by-2 matrix). As for $A$ being invertible or not, it depends on this repeated eigenvalue being zero: $\det A = \lambda^2 = 0$ iff $\lambda = 0$.

**Problem 8** *Monday 4/9*

Do problem 22 of section 6.2 in your book.

***Solution 8***

We begin by computing the eigenvalues of $A$ by solving $\det(A - \lambda I) = 0$ for $\lambda$:

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = (1-\lambda)(3-\lambda)$$

The eigenvalues are $\lambda = 1$ and $\lambda = 3$. Now, we find the corresponding eigenvectors. For $\lambda = 1$, we have $A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, so $N(A - I)$ is generated by the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, which is an eigenvector of $A$. For $\lambda = 3$, we have $A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, so $N(A - 3I)$ is generated by the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which is another eigenvector of $A$. The eigenvector matrix is

$$S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

its inverse is

$$S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

and the corresponding diagonal matrix is

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

We have $A = S\Lambda S^{-1}$, and so $A^k = S\Lambda^k S^{-1}$:

$$A^k = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^k \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1^k & 0 \\ 0 & 3^k \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3^k+1 & 3^k-1 \\ 3^k-1 & 3^k+1 \end{bmatrix}$$

**Problem 9** *Monday 4/9*

Do problem 28 of section 6.2 in your book.

***Solution 9***

Let $\mathcal{S}$ be the set of 4-by-4 matrices that are diagonalized by the same eigenvector matrix $S$, i.e., matrices $A$ such that $S^{-1}AS$ is a diagonal matrix. We want to prove that this is a subspace:
Suppose $A \in \mathcal{S}$, with $S^{-1}AS = \Lambda$ diagonal matrix, and let $c$ be a scalar. Then,

$$S^{-1}(cA)S = cS^{-1}AS = c\Lambda$$

is also a diagonal matrix. Thus, $cA$ is diagonalized by $S$, and $cA \in \mathcal{S}$.
Suppose $A_1, A_2 \in \mathcal{S}$, with $S^{-1}A_1S = \Lambda_1$ and $S^{-1}A_2S = \Lambda_2$ diagonal matrices. Then,

$$S^{-1}(A_1 + A_2)S = S^{-1}A_1S + S^{-1}A_2S = \Lambda_1 + \Lambda_2$$

is also a diagonal matrix (the sum of two diagonal matrices is diagonal). Thus, $A_1 + A_2$ is diagonalized by $S$, and $A_1 + A_2 \in \mathcal{S}$.
Alternatively, let $v_1, v_2, \ldots, v_n$ be the column vectors of $S$. Then $\mathcal{S}$ is the set of 4-by-4 matrices that have $v_1, v_2, \ldots, v_n$ as eigenvectors. But the eigenvectors of $cA$ are the same as those of $A$ (prove this!), and if $A_1, A_2$ have the same eigenvectors, then so does $A_1 + A_2$ (prove this!).
In the case that $S$ is the identity matrix, then $S^{-1}AS = I^{-1}AI = A$ must be a diagonal matrix. Thus, $\mathcal{S}$ is the space of 4-by-4 diagonal matrices, which has dimension 4.

**Problem 10** *Monday 4/9*

(a) Give an example of a $3 \times 3$ matrix $A \neq 0$ such that $A^2 \neq 0$ but $A^3 = 0$. Four your $A$ find all the eigenvalues and the eigenvectors.
(b) Now, let $B$ be a diagonalizable matrix such that there exists some positive integer $k$ such that $B^k = 0$. Prove that $B = 0$.
(c) Does part (b) contradict part (a)? Explain your answer.

***Solution 10***

(a) One such example is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Then, $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. To find the eigenvalues we solve $\det(A - \lambda I) = 0$ for $\lambda$.

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{bmatrix} = -\lambda^3$$

so $\lambda = 0$ is the only eigenvalue. There is only one eigenvector, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, which spans the nullspace of $A - 0I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.
(b) Now, let $B$ be a diagonalizable matrix such that $B^k = 0$ for some $k$. Since $B$ is diagonalizable, we can write $\Lambda^k = S^{-1}B^kS = S^{-1}0S = 0$. But because $\Lambda$ is a diagonal matrix, this implies that $\Lambda = 0$:

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} = 0 \Longrightarrow \forall i \lambda_i^k = 0 \Longrightarrow \forall i \lambda_i = 0 \Longrightarrow \Lambda = 0$$

(c)No, there is no contradiction, because $A$ in (a) was not diagonalizable (not enough independent eigenvectors)!