18.06 **QUIZ 2** April 06, 2007

Your PRINTED name is: SOLUTIONS

Please circle your recitation:

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Grading
1
2
3
4
Total:

Problem 1 *(25 points)*

(a) Compute the determinant of the matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 4 & 1 & 1 \end{pmatrix}$

(b) Compute the determinant of the matrix $B = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 4 & 1 & 1 \\ 5 & 1 & 1 & 1 \end{pmatrix}$

(c) Show that the matrix B from (b) is invertible and calculate the entry $(1, 4)$ of the inverse matrix B^{-1}.

Solution 1

(a) Using the 3×3 "big formula": 3+4+2-24-1-1=-17.

(b) $det \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 4 & 1 & 1 \\ 5 & 1 & 1 & 1 \end{pmatrix} = det \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & -4 & -4 & -9 \end{pmatrix} = 74.$

(c) Since $detB = 74 \neq 0$, B is invertible.

By cofactors, the (1,4) entry of B^{-1} is $(-1)^5 C_{4,1} = \frac{-(-17)}{74} = \frac{17}{74}$.

Problem 2 *(25 points)*

(a) Compute the projection of the vector $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ onto the column space of $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 0 & -2 \end{pmatrix}$.

(Hint: first check whether A has linearly independent columns.)

(b) Find the least-square solution $\hat{x}$ for the system

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(c) Find the projection of the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ onto the column space of $\begin{pmatrix} 10000 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 0 & -2 \end{pmatrix}$.

(Hint: No computations!)

Solution 2

(a) The third column is in the span of the first two columns. So in order to calculate the projection we need to use the matrix $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The projection is $p = B(B^T B)^{-1} B^T b = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$.

(b) $\hat{x} = (B^T B)^{-1} B^T b = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$.

(c) The columns are linearly independent so the projection is $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$.

Problem 3 *(25 points)*

Consider the basis $a_1 = (1,0,1,0)^T$, $a_2 = (1,1,1,1)^T$, $a_3 = (0,0,2,0)^T$, $a_4 = (0,0,0,2)^T$ of $\mathbb{R}^4$. Transform this basis into an orthogonal basis using the Gram-Schmidt process.

In other words, find the orthogonal basis b_1, b_2, b_3, b_4 of $\mathbb{R}^4$ such that

$$\begin{aligned}
b_1 &= a_1, \\
b_2 &= a_2 - (\text{some coefficient})\, b_1, \\
b_3 &= a_3 - (\text{some linear combination of } b_1, b_2), \\
b_4 &= a_4 - (\text{some linear combination of } b_1, b_2, b_3).
\end{aligned}$$

Solution 3

$$b_1 = a_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$b_2 = a_2 - \frac{b_1^T a_2}{b_1^T b_1} b_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$b_3 = a_3 - \frac{b_1^T a_3}{b_1^T b_1} b_1 - \frac{b_2^T a_3}{b_2^T b_2} b_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$b_4 = a_4 - \frac{b_1^T a_4}{b_1^T b_1} b_1 - \frac{b_2^T a_4}{b_2^T b_2} b_2 - \frac{b_3^T a_4}{b_3^T b_3} b_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

It was important to use b_2 and not a_2 to compute b_3, and similarly, use b_2 and b_3 and not a_2 and a_3 to compute b_4.

Problem 4 *(25 points)*

Consider the matrix $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$.

(a) Show that the columns of A are orthogonal to each other.

(b) Calculate the determinant of A.

(c) Calculate the inverse matrix A^{-1}.

Solution 4

(a) Check the 6 dot products between the columns are all zero.

(b) $det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 4.$

(c) The easiest way of computing this inverse is to use part (a):

$$A^T A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 2I$$

Thus $A^{-1} = \frac{1}{2} A^T$.