18.06 Spring 2006 - Exam 1 Review Problems

SOLUTIONS TO SELECTED PROBLEMS

1. Section 2.2, Problem 22

Answer: The solution is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ instead of $\begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$.

2. Section 2.5, Problem 10

Answer: $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}$ (invert each block).

3. Section 2.6, Problem 11

Answer: $A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}$ has $L = I$ and $D = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 7 \end{bmatrix}$;

$A = LU$ has $U = A$; $A = LDU$ has $U = D^{-1}A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

4. Section 2.7, Problem 3

Answer:

a) $((AB)^{-1})^T = (B^{-1}A^{-1})^T = (A^{-1})^T(B^{-1})^T$

b) If U is upper triangular then $(U^{-1})^T$ is **lower** triangular

5. Section 2.7, Problem 15

Answer:

a) If P sends row 1 to row 4, then P^T sends **row 4** to **row 1**.

b) $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

6. Section 3.1, Problem 18

Answer:

a) True

b) True

c) False; $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

A, B are not symmetric but $A + B$ **is** symmetric.

7. Section 3.1, Problem 29

Answer: If the 9 by 12 system $A\mathbf{x} = \mathbf{b}$ is solvable for every $\mathbf{b}$, then $\mathbf{C}(A) = \mathbb{R}^9$ (every $\mathbf{b}$ is in the column space).

8. Section 3.2, Problem 5

Answer:

$$A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}.$$

9. Section 3.2, Problem 7

Answer:

a) The nullspace of A is the plane $-x + 3y + 5z = 0$; it contains all vectors of the form $(3y + 5z, y, z)$.

b) The nullsapce of B is the line through $(3, 1, 0)$.

10. Section 3.3, Problem 9

Answer: If A is an m by n matrix with $r = 1$, its columns are multiples of one column and its rows are multiples of one row. The column space is a **line** in $\mathbb{R}^m$. The nullspace is a **plane** in $\mathbb{R}^n$. Also the column space of A^T is a **line** in $\mathbb{R}^n$.

11. Section 3.3, Problem 26

Answer:

a) If $c = 1$, then $R = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has x_2, x_3, x_4 free.

If $c \neq 1$, $R = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has x_3, x_4 free.

Special solutions $N = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for $c = 1$ and $N = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ for $c \neq 1$.

b) If $c = 1$, $R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and x_1 is free; if $c = 2$, $R = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ and x_2 is free; $R = I$ if $c \neq 1, 2$. Special solutions $N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for $c = 1$; $N = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for $c = 2$.

12. Section 3.4, Problem 3

Answer: $x_{complete} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$.

13. Section 3.4, Problem 32

Answer: $A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

For $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 5 \end{bmatrix}$, the solution is $\mathbf{x} = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}$. For $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, there is no solution.

14. Section 3.5, Problem 5

Answer:

a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -\frac{18}{5} \end{bmatrix}$

The matrix is invertible, so the columns are independent.

b) $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$

The matrix is **not** invertible ($(1,1,1)^T$ is in the nullspace), so the columns are dependent.

15. Section 3.5, Problem 10

Answer: The plane is the nullspace of $A = \begin{bmatrix} 1 & 2 & -3 & -1 \end{bmatrix}$. There are three free variables so you can find at most three independent vectors in the plane. For example $\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

16. Section 3.5, Problem 17

Answer: These bases are not unique

a) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

c) $\begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$

d) Column space: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; Nullspace: $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$