18.06 Spring 2006 - Problem Set 7

SOLUTIONS TO SELECTED PROBLEMS

1. Section 6.1, Problem 2

Answer: A has $\lambda_1 = -1$ and $\lambda_2 = 5$ with eigenvectors $\mathbf{x_1} = (-2, 1)$ and $\mathbf{x_2} = (1, 1)$. The matrix $A + I$ has the same eigenvectors with eigenvalues increased by 1: $\lambda_1 = 0$ and $\lambda_2 = 6$.

2. Section 6.1, Problem 12

Answer: P has $\lambda = 1, 0, 1$ with eigenvectors $(1, 2, 0), (2, -1, 0), (0, 0, 1)$. $P^{100} = P$ so P^{100} has the same eigenvalues and eigenvectors.

An eigenvector with no zero components is $(1, 2, 0) + (0, 0, 1) = (1, 2, 1)$ which has $\lambda = 1$.

3. Section 6.1, Problem 22

Answer: A and A^T have the same eigenvalues because $\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - (\lambda I)^T) = \det(A^T - \lambda I)$.

$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $A^T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have different eigenvectors.

4. Section 6.1, Problem 28

Answer: $\text{rank}(A) = 1$, with $\lambda = 0, 0, 0, 4$. $\text{rank}(C) = 2$, with $\lambda = 0, 0, 2, 2$.

5. Section 6.2, Problem 3

Answer:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$$

6. Section 6.2, Problem 15

Answer: (No explanation necessary.)

a) True; all eigenvalues are non-zero.

b) False; may have 2 or 3 independent eigenvectors.

c) False; may have 2 or 3 independent eigenvectors.

7. Section 6.2, Problem 22

Answer:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

$$A^k = \tfrac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} 3^k+1 & 3^k-1 \\ 3^k-1 & 3^k+1 \end{bmatrix}.$$

8. Section 6.2, Problem 29

Answer: If A has columns $\mathbf{x_1}, \ldots, \mathbf{x_n}$, then $A^2 = A$ means that $A\mathbf{x_i} = \mathbf{x_i}$ for every $\mathbf{x_i}$. All vectors in the column space are eigenvectors with $\lambda = 1$. Always the nullspace has $\lambda = 0$.

9. Section 8.3, Problem 12

Answer: $.2, .3, .5$ as the last row makes A Markov and symmetric. When A is Markov and symmetric, each row adds to 1 so $(1, 1, 1, 1)$ is an eigenvector of A.

10. Section 10.2, Problem 2

Answer:

$A^H A = \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & 1-i & 2 \end{bmatrix}$ and $A^H A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ are both **Hermitian** matrices.

11. Section 10.2, Problem 8

Answer: P is orthogonal, invertible, unitary and factorizable into QR.