18.06 Spring 2006 - Problem Set 4

SOLUTIONS TO SELECTED PROBLEMS

1. Section 3.6, Problem 15

Answer: If the first two rows of A are exchanged then the **row space** and **nullspace** stay the same.

$\begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ is in the new column space.

2. Section 3.6, Problem 23

Answer:

Basis for row space: $\begin{bmatrix} 3 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$.

Basis for column space: $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$.

A is 3×3 but rank of A is 2, so A is not invertible.

3. Section 3.6, Problem 25

Answer:

a) True; A and A^T have the same rank.

b) False; $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$

c) False; $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

d) True; $\mathbf{C}(A) = \mathbf{C}(-A)$ and $\mathbf{RS}(A) = \mathbf{RS}(-A)$.

4. Section 4.1, Problem 4

Answer: If $AB = 0$ then the columns of B are in the **nullspace** of A. The rows of A are in the **left nullspace** of B.

A and B can't be 3 by 3 matrices of rank 2 because that would mean the four subspaces of A and B all have dimension 2.

5. Section 4.1, Problem 5

Answer:

a) If $A\mathbf{x} = \mathbf{b}$ has a solution and $A^T\mathbf{y} = \mathbf{0}$, then $\mathbf{y}$ is perpendicular to $\mathbf{b}$.

b) If $A^T\mathbf{y} = \mathbf{c}$ has a solution and $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ is perpendicular to $\mathbf{c}$.

6. Section 4.1, Problem 17

Answer:

If $\mathbf{S}$ is the subspace of $\mathbb{R}^3$ containing only the zero vector, then $\mathbf{S}^\perp = \mathbb{R}^3$.

If $\mathbf{S}$ is spanned by $(1, 1, 1)$, $\mathbf{S}^\perp$ is spanned by $(1, -1, 0), (1, 0, -1)$.

If $\mathbf{S}$ is spanned by $(2, 0, 0), (0, 0, 3)$, $\mathbf{S}^\perp$ is spanned by $(0, 1, 0)$.

(Use the basis vectors of $\mathbf{S}$ as rows of a matrix. Then a basis for $\mathbf{S}^\perp$ is the basis of the nullspace.)

7. Section 4.1, Problem 22

Answer: $\mathbf{P}$ is the nullspace of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$. A basis for $\mathbf{P}^\perp$ is $(1, 1, 1, 1)$.

8. Section 4.2, Problem 13

$$\textit{Answer:}\ P\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

9. Section 4.2, Problem 17

Answer: If $P^2 = P$, then $(I - P)^2 = I^2 - 2P + P^2 = I - P$. When P projects onto the column space of A, $I - P$ projects onto the **left nullspace** of A. ($(I - P)\mathbf{b} = \mathbf{b} - P\mathbf{b} = \mathbf{e}$.)

10. Section 4.2, Problem 19

Answer: For any choice, say $(1, 1, 0), (2, 0, 1)$, the matrix P is $\begin{bmatrix} \frac{5}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

11. Section 4.2, Problem 27

Answer: **If $\mathbf{A^T A x = 0}$, then $\mathbf{Ax = 0}$**. The vector $A\mathbf{x}$ is in the nullspace of $\mathbf{A^T}$. $A\mathbf{x}$ is always in the column space of $\mathbf{A}$. To be in both perpendicular spaces, $A\mathbf{x}$ must be zero.

12. COMPUTER PROBLEM:

Answer: For a 5 by 5 matrix, there are $5! = 120$ terms; 60 are positive and 60 are negative. For a 6 by 6 matrix there are $6! = 720$ terms; 360 are positive and 360 are negative.

5 by 5 case in Maple:

```
> with(linalg):

> A:=<<a|b|c|d|e>,<f|g|h|i|j>,<k|l|m|n|o>,<p|q|r|s|t>,<u|v|w|x|y>>;
                              [a    b    c    d    e]
                              [                    ]
                              [f    g    h    i    j]
                              [                    ]
                         A := [k    l    m    n    o]
                              [                    ]
                              [p    q    r    s    t]
                              [                    ]
                              [u    v    w    x    y]
```

```
> det(A);
a g m s y - a g m t x - a g r n y + a g r o x + a g w n t - a g w o s

- a l h s y + a l h t x + a l r i y - a l r j x - a l w i t + a l w j s

+ a q h n y - a q h o x - a q m i y + a q m j x + a q w i o - a q w j n

- a v h n t + a v h o s + a v m i t - a v m j s - a v r i o + a v r j n

- f b m s y + f b m t x + f b r n y - f b r o x - f b w n t + f b w o s

+ f l c s y - f l c t x - f l r d y + f l r e x + f l w d t - f l w e s

- f q c n y + f q c o x + f q m d y - f q m e x - f q w d o + f q w e n

+ f v c n t - f v c o s - f v m d t + f v m e s + f v r d o - f v r e n

+ k b h s y - k b h t x - k b r i y + k b r j x + k b w i t - k b w j s

- k g c s y + k g c t x + k g r d y - k g r e x - k g w d t + k g w e s

+ k q c i y - k q c j x - k q h d y + k q h e x + k q w d j - k q w e i
```

- k v c i t + k v c j s + k v h d t - k v h e s - k v r d j + k v r e i

- p b h n y + p b h o x + p b m i y - p b m j x - p b w i o + p b w j n

+ p g c n y - p g c o x - p g m d y + p g m e x + p g w d o - p g w e n

- p l c i y + p l c j x + p l h d y - p l h e x - p l w d j + p l w e i

+ p v c i o - p v c j n - p v h d o + p v h e n + p v m d j - p v m e i

+ u b h n t - u b h o s - u b m i t + u b m j s + u b r i o - u b r j n

- u g c n t + u g c o s + u g m d t - u g m e s - u g r d o + u g r e n

+ u l c i t - u l c j s - u l h d t + u l h e s + u l r d j - u l r e i

- u q c i o + u q c j n + u q h d o - u q h e n - u q m d j + u q m e i