18.06 Spring 2006 - Problem Set 3

SOLUTIONS TO SELECTED PROBLEMS

1. Section 3.2, Problem 15

Answer: Suppose an m by n matrix has r pivots. The number of special solutions is $\mathbf{n} - \mathbf{r}$. The nullspace contains only $\mathbf{x} = \mathbf{0}$ when $\mathbf{r} = \mathbf{n}$. The column is space is all of $\mathbb{R}^m$ when $\mathbf{r} = \mathbf{m}$.

2. Section 3.2, Problem 18

Answer: All points on the plane have the form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

3. Section 3.2, Problem 23

Answer: $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$

4. Section 3.2, Problem 25

Answer: $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$

5. Section 3.2, Problem 27

Answer: If nullspace = column space then $n - r = r$ (there are r pivots). For $n = 3$, $3 = 2r$ is impossible.

6. Section 3.2, Problem 28

Answer: If $AB = 0$ then the column space of B is contained in the **nullspace** of A. An example of such A, B is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

7. Section 3.3, Problem 8

Answer:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}, B = \begin{bmatrix} 2 & 6 & -3 \\ 1 & 3 & -\frac{3}{2} \\ 2 & 6 & -3 \end{bmatrix}, M = \begin{bmatrix} a & b \\ c & \frac{cb}{a} \end{bmatrix}$$

8. Section 3.3, Problem 17

Answer:

a) The jth column of AB is A times the jth column of B. If column j of B is a combination of the previous columns of B then AB will be a combination of the previous columns of AB.

b) rank($A_1 B$)= 1 for $A_1 = I$; rank($A_2 B$)=0 for $A_2 = 0$ or $A_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

9. Section 3.4, Problem 5

Answer: Elimination on the augmented matrix:

$$\begin{bmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 1 & 0 & b_3 - 4b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{bmatrix}$$

This is solvable if $b_3 = 2b_1 + b_2$. If this condition holds, then $y = b_2 - 2b_1$ and $x = b_1 - 2y + 2z = b_1 - 2(b_2 - 2b_1) + 2z = 5b_1 - 2b_2 + 2z$. z is a free variable, so letting $z = 0$, we get that the particular solution is $\begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix}$ and the special solution is $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

The complete solution is $\mathbf{x} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

10. Section 3.4, Problem 22

Answer: If $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions then we can find $\mathbf{x_1}, \mathbf{x_2}$ such that $A\mathbf{x_1} = \mathbf{b}$ and $A\mathbf{x_2} = \mathbf{b}$. Then $\mathbf{x_1} - \mathbf{x_2}$ is in the nullspace of A. We can add $\mathbf{x_1} - \mathbf{x_2}$ to any solution of $A\mathbf{x} = \mathbf{B}$ to get a new solution; hence $A\mathbf{x} = \mathbf{B}$ cannot have exactly one solution.

$A\mathbf{x} = \mathbf{B}$ will have no solution if $\mathbf{B}$ is not in the column space of A.

11. Section 3.4, Problem 24

Answer:

a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 \end{bmatrix}$

c) $A = 0$ or any matrix with $r < m$ and $r < n$.

d) Any invertible matrix.

12. Section 3.4, Problem 31

Answer:

$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}$ or any 3×2 matrix with rank 2 and second column $= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

B cannot exist. 2 equations in 3 unknowns cannot have a unique solution.

13. Section 3.4, Problem 33

Answer: $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$.

The particular solution is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So we have $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$ first column of $A = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. The special solution, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, is in the nullspace of A.

This gives $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$ the second column of $A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.