18.06 - Final Exam, Monday May 16th, 2005

SOLUTIONS

1. (a) We want the coordinates $(a_{i1}, \ldots, a_{in})$ of P_i to satisfy the equation $c_1 x_1 + \ldots + c_n x_n = 1$. Thus the system of equations is $Ac = \texttt{ones}$:

$$\begin{aligned} c_1 a_{11} + c_2 a_{12} + \ldots + c_n a_{1n} &= 1 \\ c_1 a_{21} + c_2 a_{22} + \ldots + c_n a_{2n} &= 1 \\ &\ldots \\ c_1 a_{n1} + c_2 a_{n2} + \ldots + c_n a_{nn} &= 1 \end{aligned}$$

(b) There is no plane of the given form, if one of the points P_i is the origin. More than one plane contains the P_i's if the three points are on a line not through the origin.

(c) There is not a unique solution precisely when $\det A = 0$. This means geometrically that the points P_i lie in an $(n-1)-$dimensional subspace of $\mathbf{R}^n$.

2. (a) Subtracting the first row from the second, we find the matrix

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(In the row reduced echelon form R, the 5 changes to 0.) The pivot variables are the first and the last, while x_2, x_3, x_4 are the free variables. Thus the "special solutions" to $Ax = 0$ are

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

(b) and (c) We need to prove that these three vectors are *linearly independent* and they *span the nullspace*. By considering the second, third and fourth coordinates, a combination of the vectors adding to zero must have zero coefficients. The vectors span the nullspace, since the dimension of the nullspace is three (note that the rank of the matrix A is 2).

3. (a) If $Ax = b$ has no solution, the column space of A must have dimension less than m. The rank is $r < m$. Since $A^{\mathrm{T}} y = c$ has exactly one solution, the columns of A^{T} are independent. This means that the rank of A^{T} is $r = m$. This contradiction proves that we cannot find A, b and c.

(b) We need to check two statements: the vector $b - p$ is orthogonal to the space generated by $a_1, \ldots, a_n$ and the vector p lies in that subspace. The first condition we check by seeing if the scalar products $a_1 \cdot (b-p), \ldots, a_n \cdot (b-p)$ all equal zero. The second condition we check by considering the $(n+1) \times m$ matrix whose first n rows are the coordinates of the a_i's and whose last row consists of the coordinates of p. The vector p is in the span of the a_i's if and only if the last row becomes zero in elimination.

4. (a) To compute the determinant, subtract the second row from all the other rows:

$$\det B = \det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1\,.$$

(b) $\lambda = 1, 1, 1, 1, 6$. Since $A - I$ has all equal rows, it has rank one. It follows that it has four zero eigenvalues. The eigenvalues of A are the eigenvalues of $A - I$ increased by one, so A has the eigenvalue 1 with multiplicity four. The trace of A equals 10 so $10 - 4 = 6$ is the other eigenvalue.

(c) A is symmetric, and thus so is A^{-1}. The cofactor formula gives:

$$(A^{-1})_{13} = (-1)^{1+3} \frac{\det B}{\det A}\,,$$

and $\det A = 6$ since it equals the product of the eigenvalues of A. We conclude that the $(1,3)$ and the $(3,1)$ entries of A^{-1} are both equal to $-1/6$.

5. (a) The answer is

$$A = \begin{bmatrix} 2 & 6 \\ -1 & 7 \end{bmatrix}.$$

Reason:

$$\begin{bmatrix} 2 & 6 \\ a & b \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 3a+b \end{bmatrix} = \lambda_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

We deduce that $\lambda_1 = 4$ and $3a + b = 4$. Similarly, since x_2 is an eigenvector we have

$$\begin{bmatrix} 2 & 6 \\ a & b \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 2a+b \end{bmatrix} = \lambda_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

We deduce that $\lambda_2 = 5$ and therefore that $2a+b = 5$. We conclude that $\boldsymbol{a = -1}$ **and** $\boldsymbol{b = 7}$**.**

(b) $B = S\Lambda S^{-1}$, where the columns of S are the vectors x_1 and x_2, and Λ is the diagonal matrix with entries 1 and 0:

$$B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}.$$

Then $\Lambda^{10} = \Lambda$ and therefore $B^{10} = S\Lambda^{10}S^{-1} = S\Lambda S^{-1} = B$.

6. (a) We would solve the equations

$$\begin{aligned} c_0 + c_1 + c_2 + c_3 &= y_1 \\ c_0 + i\,c_1 - c_2 - i\,c_3 &= y_2 \\ c_0 - c_1 + c_2 - c_3 &= y_3 \\ c_0 - i\,c_1 - c_2 + i\,c_3 &= y_4\,. \end{aligned}$$

and the matrix of coefficients is

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1^2 & i^2 & i^4 & i^6 \\ 1^3 & i^3 & i^6 & i^9 \end{bmatrix}.$$

(b) F has *orthogonal columns* and it is *symmetric*. It is also a Vandermonde matrix: each column consists of the first four powers of a number (starting from the zero-th power).

(c) Since the columns of F are orthogonal and non-zero, the matrix is invertible. Its inverse is $\overline{F}/4$. The determinant of this Vandermonde matrix is equal to the product of the differences of $1, i, i^2, i^3$:

$$\det F = (i-1)(-1-1)(-1-i)(-i-1)(-i-i)(-i+1) = -16i\,.$$

7. (a) The seven eigenvalues of P are $1, 1, 1, 1, 0, 0, 0$.

(b) The eigenvectors with eigenvalue 1 are the non-zero vectors in S. The eigenvectors with eigenvalue 0 are the non-zero vectors in the orthogonal complement of S.

(c) The solution $u(t)$ to the differential equation has the form

$$u(t) = v_1 e^{-t} + v_2\,,$$

where v_1 is in S and v_2 is in the orthogonal complement of S. Then $u(\infty) = v_2$, which is the projection of $u(0)$ onto the orthogonal complement of S.

8. (a) The required matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) Since B is block diagonal, its eigenvalues are the eigenvalues of the diagonal blocks. In our case, the two blocks are the same and the eigenvalues of each block are 3 and 1. Thus the eigenvalues of B are $3, 3, 1, 1$.

(c) Since P is a permutation matrix, it is orthogonal and therefore $P^{\mathrm{T}} = P^{-1}$. The matrix B is thus similar to the matrix A and we conclude that A and B have the same eigenvalues.

The function of u, v, w, z which is positive except if $u = v = w = z = 0$ is thus

$$\begin{bmatrix} u & v & w & z \end{bmatrix} A \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = 2\left(u^2 + v^2 + w^2 + z^2 - uw - vz\right).$$

9. (a) The vectors orthogonal to the nullspace of A are the rows of A. Since we know that the matrix A is singular and it is clearly not rank one, it follows that the rank of A is two. The first two rows are independent and therefore the orthogonal complement of the nullspace of A is spanned by the two vectors

$$\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}.$$

(b) We get the vectors

$$\frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(c) The "reduced" LU decomposition, from ignoring the zero row in U, is

Answer $$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 2 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \end{bmatrix}.$$

Here are the details: Starting elimination we find

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 2 & 2 & 6 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \\ 0 & -5 & -10 \end{bmatrix}, \tag{1}$$

and proceeding further we find

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{5}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \\ 0 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \\ 0 & 0 & 0 \end{bmatrix}.$$

Collecting all the information together we obtain

$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 2 & 6 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{5}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \\ 0 & 0 & 0 \end{bmatrix},$$

and multiplying the first two matrices on the right-hand side we deduce that

$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 2 & 6 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & \frac{5}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the last row of the last matrix is all zero, we conclude that

$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 2 & 6 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 2 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \end{bmatrix}.$$

This is the "reduced" LU factorization of A. Multiplying columns of L by rows of U, this is

$$A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \frac{5}{4} \end{bmatrix} \begin{bmatrix} 0 & -4 & -8 \end{bmatrix}.$$

10. (a) The eigenvalues of $A^{\mathrm{T}}A$ are the same as the eigenvalues of $\Sigma^{\mathrm{T}}\Sigma$ which is the 4 by 4 diagonal matrix with entries 1, 16, 0, 0 along the diagonal.

(b) The nullspace $\boldsymbol{N}(A)$ is spanned by the *last two columns* of V.

(c) The column space of A is spanned by the first two columns of U.

(d) A singular value decomposition of $-A^{\mathrm{T}}$ is $-A^{\mathrm{T}} = (-V)\Sigma^{\mathrm{T}}U^{\mathrm{T}}$.