Exam 2, Friday April 1st, 2005

Solutions

Question 1. The vector a_1 can be any non-zero positive multiple of q_1. The vector a_2 can be any multiple of q_1 plus any non-zero positive multiple of q_2:

$$\begin{array}{rcl} a_1 & = & cq_1 \\ a_2 & = & c_1q_1 + c_2q_2 \end{array} \quad , \text{ with } c, c_1 > 0.$$

Question 2. We want to find the least squares solution to the equation

$$ax = b$$

and we know that it is enough to multiply both sides by a^T and solve the resulting system:

$$a^T ax = a^T b \qquad \Longrightarrow \qquad x = \frac{a \cdot b}{a \cdot a}$$

Question 3. The vectors $(-1, 1, 0)^T$ and $(-1, 0, 1)^T$ form a basis for the subspace $x + y + z = 0$. Let A be the matrix whose columns are the two vectors found above. Thus the projection matrix P onto the subspace $x + y + z = 0$ is

$$\begin{aligned} P &= A\left(A^T A\right)^{-1} A^T = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \\ &= \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \\ &= \frac{1}{3} \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \\ &= \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \end{aligned}$$

The projection of $(1, 2, 6)^T$ onto the plane $x + y + z = 0$ is thus simply

$$p = P \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$$

Question 4. Looking at the first row of A we deduce that

$$\det A = \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 1 & 3 & 9 \end{pmatrix} = 9 - 6 = 3$$

Of course, $\det(A^{-1}) = \frac{1}{3}$. Finally

$$\left(A^{-1}\right)_{12} = \frac{-C_{21}}{\det A} = \frac{-1}{3} \det \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \\ 3 & 1 & 9 \end{pmatrix} = \frac{-(-9)}{3} = 3$$

Question 5. (a) The column space of QQ^T is at most two dimensional, since the matrix QQ^T is 4×4, it cannot have rank four. Thus $\det QQ^T = 0$.

Similarly, the matrix $[\ Q\ Q\ ]$ has dependent columns, and therefore $\det[\ Q\ Q\ ] = 0$.

(b) Using the projection formula,

$$p = Q(Q^TQ)^{-1}Q^Tb = QIQ^Tb = QQ^Tb$$

(c) The error vector $e = b - p$ is contained in the left null-space of Q, the nullspace of Q^T. To check this, we compute

$$Q^Te = Q^T\left(b - QQ^Tb\right) = Q^Tb - Q^TQQ^Tb = Q^Tb - IQ^Tb = 0$$

Question 6. The product P_2P_1 is projection onto the column space of P_1, followed by the projection onto the column space of P_2. Since the column space of P_2 contains the column space of P_1, the second projection does not change the vectors anymore. Thus

$$P_2P_1 = P_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 2 & 0 & 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$$