18.06 Professor A.J. de Jong Exam 1 Solutions March 3, 2003

1 (30 pts.)

(a) (10 pts)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -5 & -4 & -3 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 15 & 40 \\ -15 & -20 \end{pmatrix}$$

(b) (20 pts)

(10 pts) Computing R:

$$U = \begin{pmatrix} 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 7 & 5 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 & -9 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & -9 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = R$$

(8 pts) Special solutions to $Ux = 0$:

$$x_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 9 \\ 0 \\ -7 \\ 1 \\ 0 \end{pmatrix}$$

(2 pts) General solutions to $Ux = 0$:

$$x = a \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 9 \\ 0 \\ -7 \\ 1 \\ 0 \end{pmatrix}, \; a, b \in \mathbb{R}$$

2 (35 pts.)

(a) (25 pts) Find the complete solution to the equation $Ax = b$ using the algorithm described in class and in the book.

$$(A \mid b) = \begin{pmatrix} 1 & 0 & 0 & 4 & | & 2 \\ 1 & 0 & 1 & 0 & | & 0 \\ 1 & 1 & 0 & 0 & | & 0 \\ 1 & 2 & 3 & t & | & s \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 4 & | & 2 \\ 0 & 0 & 1 & -4 & | & -2 \\ 0 & 1 & 0 & -4 & | & -2 \\ 0 & 2 & 3 & t-4 & | & s-2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 4 & | & 2 \\ 0 & 1 & 0 & -4 & | & -2 \\ 0 & 0 & 1 & -4 & | & -2 \\ 0 & 0 & 3 & t+4 & | & s+2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 4 & | & 2 \\ 0 & 1 & 0 & -4 & | & -2 \\ 0 & 0 & 1 & -4 & | & -2 \\ 0 & 0 & 0 & t+16 & | & s+8 \end{pmatrix}$$

Therefore, when $t = -16$ and $s \neq -8$ there are no solutions.

When $t \neq -16$ there is a unique solution:

$$x = \begin{pmatrix} 2 - 4\frac{s+8}{t+16} \\ -2 + 4\frac{s+8}{t+16} \\ -2 + 4\frac{s+8}{t+16} \\ \frac{s+8}{t+16} \end{pmatrix}.$$

When $t = -16$ and $s = -8$, there are infinitely many solutions:

$$x = \begin{pmatrix} 2 - 4a \\ -2 + 4a \\ -2 + 4a \\ a \end{pmatrix}, \quad \text{for all} \quad a \in \mathbb{R}.$$

(b) First part (5 pts)

From the previous computation, column vectors of A are linearly independent when $t \neq -16$ (all pivots are nonzero).

When $t = -16$, column vectors are linearly dependent:

$$4 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 4 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} - 4 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} - 1 \cdot \begin{pmatrix} 4 \\ 0 \\ 0 \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Second part (5 pts)

When $s \neq -8$ column vectors are linearly independent. Indeed, after swapping the last column of A with b, in the computation in (a) all pivots are nonzero.

When $s = -8$, column vectors are linearly dependent:

$$2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} - 1 \cdot \begin{pmatrix} 4 \\ 0 \\ 0 \\ -8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

3 (35 pts.)

(a) Row operations (15 pts) :

- add 10 times row 2 to row 3
- add 11 times row 3 to row 4
- swap rows 1 and 2
- add 4 times row 4 to row 1

"Why the upper left corner of A is zero" question. (5 pts)

Answer: The upper right corner is zero since otherwise we would never have needed to swap rows 1 and 2.

(b) (15 pts):

The matrix U corresponds to the upward elimination. So we get

$$U = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix P corresponds to the permutation of the first two rows. So we have

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{T} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix L corresponds the first two elimination steps. So we have

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 10 & 1 & 0 \\ 0 & 110 & 11 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -10 & 1 & 0 \\ 0 & 0 & -11 & 1 \end{bmatrix}.$$