18.06 Exam 2 #1 Solutions

1. The row echelon form of A is $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So we find that a basis for $R(A^T)$ is $\{(1,2,-1,4),(0,1,-2,3)\}$, and a basis for $N(A)$ is $\{(-3,2,1,0),(2,-3,0,1)\}$. Similarly, row echelon form of A^T is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So a basis for $C(A)$ is $\{(1,0,-1),(0,1,2)\}$ and a basis for $N(A^T)$ is $\{(1,-2,1)\}$.

2. a) Using the row operation $R4 - R1$ gives

$$\begin{vmatrix} -1 & 2 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & 1 & 2 & 0 \\ -1 & -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & 1 & 2 & 0 \\ -0 & -3 & 0 & 0 \end{vmatrix}.$$

Using a cofactor expansion about the fourth column gives

$$\begin{aligned} \begin{vmatrix} -1 & 2 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & 1 & 2 & 0 \\ -0 & -3 & 0 & 0 \end{vmatrix} &= (-1)\begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \\ 0 & 3 & 0 \end{vmatrix} \\ &= (-1)(3)\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = -12. \end{aligned}$$

(here, we computed the 3×3 determinant by expanding about the third row.

b) Using Cramer's rule,

$$A^{-1}(1,4) = \frac{C_{4,1}}{\det(A)} = \frac{(-1)^{4+1}\begin{vmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix}}{\det(A)} = \frac{-3}{-12} = \frac{1}{4}$$

c)

$$\begin{aligned} \det(2A^2A^T(A^{-1})^3) &= \det(2I)\det(A)^2\det(A^T)\det(A^{-1})^3 \\ &= 2^4 \cdot \det(A)^2\det(A)\det(A)^{-3} = 16 \end{aligned}$$

3. A basis for the space in question is $\{(1,1,0,0),(2,0,1,0),(-1,0,0,1)\}$. To get a orthogonal basis, we need to do the Gram-Schmidt algorithm. Start with $v_1 = (1,1,0,0), v_2 = (2,0,1,0), v_3 = (-1,0,0,1)$,

$$\begin{aligned} \tilde{v_1} &= v_1 = (1,1,0,0) \\ \tilde{v_2} &= v_2 - \frac{(\tilde{v_1}, v_2)}{|\tilde{v_1}|^2}\tilde{v_1} = (1,-1,1,0) \\ \tilde{v_3} &= v_3 - \frac{(\tilde{v_1}, v_3)}{|\tilde{v_1}|^2}\tilde{v_1} - \frac{(\tilde{v_2}, v_3)}{|\tilde{v_2}|^2}\tilde{v_2} = (-\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, 1) \end{aligned}$$

Then $\{\tilde{v_1}, \tilde{v_2}, \tilde{v_3}\}$ is a set of orthogonal basis, to make them orthonormal, just multiply each $\tilde{v_i}$ by the reciprocal of its norm. So an orthonormal basis for the subspace in the problem is $\{\frac{\tilde{v_1}}{|\tilde{v_1}|}, \frac{\tilde{v_2}}{|\tilde{v_2}|}, \frac{\tilde{v_3}}{|\tilde{v_3}|}\} = \{\frac{1}{\sqrt{2}}(1,1,0,0), \frac{1}{\sqrt{3}}(1,-1,1,0), \frac{1}{\sqrt{42}}(-1,1,2,6)\}$.

4. a) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{bmatrix}, \hat{x} = \begin{bmatrix} C \\ D \end{bmatrix}, b = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, $\hat{x}$ is the solution to the linear equation $A^T A\hat{x} = A^T b$, and the least squares line is $y = C + Dx$.

   b) $P = B(B^T B)^{-1} B^T$.