18.06 Solutions to Midterm Exam 2, Spring, 2001

1. (40pts.) Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & -1 \\ 9 & 5 & 1 \\ 9 & 8 & 7 \end{pmatrix}$$

(a) Find the rank of A.

- After doing row operations on the matrix A, we obtain

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since there are two non-zero pivots, A has rank 2.

(b) Find a basis for the row space of A, and find a basis for the nullspace of A. What is the dimension of the nullspace of A?

- A basis for the row space of A is given by the two pivot rows: the first two rows, since we did not have to exchange rows during row operations; $\{(1,0,-1),(3,1,-1)\}$.

The solutions of $A\mathbf{x} = \mathbf{0}$ are given by $\mathbf{x} = \alpha(1,-2,1)^T$ where $\alpha \in \mathbb{R}$. Hence, a basis for the nullspace is $(1,-2,1)^T$. Since the basis of the nullspace contains one vector, dim $N(A) = 1$.

(c) What can you say about the relation between the rank and the dimension of the nullspace of A?

- dim $N(A)$+rk(A) = number of columns of A. Here, 1+2=3.

(d) Verify that all vectors in your basis of the nullspace are orthogonal to all vectors in your basis of the row space.

- $$(1,0,-1).(1,-2,1) = 0, \quad \text{and} \quad (3,1,-1).(1,-2,1) = 0.$$

2. (30pts.) Let $a, b \in \mathbb{R}$, and let

$$A = \begin{pmatrix} 1 & 2 & 3 & a \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & b \end{pmatrix}.$$

(a) What are the dimensions of the four subspaces associated with the matrix A? This will of course depend on the values of a and b, and you should distinguish all different cases.

- After doing row operations on the matrix A, we find

$$A = \begin{pmatrix} 1 & 2 & 3 & a \\ 0 & 2 & 4 & a \\ 0 & 0 & 0 & a-2b \end{pmatrix}.$$

If $a = 2b$, then there are two non-zero pivots, and so $\text{rk}(A) = \dim \text{col}(A) = \dim \text{row}(A) = 2$. Also, $\dim \text{null}(A) = 4 - 2 = 2$ and $\dim \text{left-null}(A) = 3 - 2 = 1$.

If $a \neq 2b$, then there are three non-zero pivots, and so $\text{rk}(A) = \dim \text{col}(A) = \dim \text{row}(A) = 3$. Also, $\dim \text{null}(A) = 4 - 3 = 1$ and $\dim \text{left-null}(A) = 3 - 3 = 0$.

(b) For $a = b = 1$, give a basis for the column space of A. Is this also a basis for $\mathbb{R}^3$? Justify your answer.

- A basis is given by the columns of A which lead to non-zero pivots,

$$\text{basis} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \right\}$$

Since $\mathbb{R}^3$ has dimension 3, and these are 3 linearly independent vectors in $\mathbb{R}^3$, they are indeed are basis for $\mathbb{R}^3$.

3. (30pts.) An experiment at the seven times $t = -3, -2, -1, 0, 1, 2, 3$ yields the consistent result $b = 0$, except at the last time ($t = 3$), when we get $b = 28$. We want the best straight line $b = C + Dt$ to fit these seven data points by least squares.

(a) Write down the equation $A\mathbf{x} = \mathbf{b}$ with unknowns C and D that would be solved if a straight line exactly fit the data.

- $$\begin{pmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 28 \end{pmatrix}$$

(b) Use the method of least squares to find the best fit values for C and D.

- $A^T A = \begin{pmatrix} 7 & 0 \\ 0 & 28 \end{pmatrix}$, and $A^T\mathbf{b} = \begin{pmatrix} 28 \\ 84 \end{pmatrix}$. Solving $(A^T A)\mathbf{x} = (A^T\mathbf{b})$, we find that $\mathbf{x}_{\text{sol}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

(c) This problem is really that of projecting the vector $\mathbf{b} = (0, 0, 0, 0, 0, 0, 28)^T$ onto a certain subspace. Give a basis for that subspace, and give the projection $\mathbf{p}$ of $\mathbf{b}$ onto that subspace.

- A basis is given by the columns of A, $\{(1, 1, 1, 1, 1, 1, 1)^T, (-3, -2, -1, 0, 1, 2, 3)^T\}$. The projection is given by

$$\mathbf{p} = A\mathbf{x}_{\text{sol}} = \begin{pmatrix} -5 \\ -2 \\ 1 \\ 3 \\ 7 \\ 10 \\ 13 \end{pmatrix}.$$