18.06 Midterm Exam 1, Spring, 2001

Name	Optional Code
Recitation Instructor	Email Address
Recitation Time	

This midterm is closed book and closed notes. No calculators, laptops, cell phones or other electronic devices may be used during the exam.

There are 3 problems. Good luck.

1. (20pts.) Find a general formula for the solutions of the following linear system of equations,

• The augmented matrix is

$$\left(\begin{array}{ccc|c}
-1 & 3 & 0 & 2 & 1 \\
4 & -12 & 2 & -4 & -4 \\
-7 & 21 & 2 & 18 & 7
\end{array}\right).$$

The corresponding row reduced matrix is

$$\left(\begin{array}{ccc|c} -1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

If we let $x_4 = b$ and $x_2 = a$, then $x_3 = -2b$ and $x_1 = -1 + 3a + 2b$. The general solution thus takes the form

$$\mathbf{x} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

2. (40pts.) Let
$$A = \begin{pmatrix} 1 & 1 & b \\ a & b & b - a \\ 1 & 1 & 0 \end{pmatrix}$$
.

- (a) For a = 2 and b = 1, find the inverse of A.
- (b) For which values of a and b is the matrix A not invertible, i.e. it has less than three pivots?
- (c) For what values of a and b is the rank of A equal to a? For what values is it equal to a, equal to a?
- (d) For a = b = 2, describe the nullspace of A.

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(a) The augmented matrix is

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

We perform row operations to obtain

$$\left(\begin{array}{ccc|ccc|c}
1 & 0 & 0 & 1 & 1 & -2 \\
0 & 1 & 0 & -1 & -1 & 3 \\
0 & 0 & 1 & 1 & 0 & -1
\end{array}\right).$$

The matrix on the right is the inverse of A.

(b) If we do row operations on the matrix

$$\left(\begin{array}{ccc}
1 & 1 & b \\
a & b & b-a \\
1 & 1 & 0
\end{array}\right)$$

we obtain

$$\left(\begin{array}{cccc} 1 & 1 & b \\ 0 & b-a & b-a-ab \\ 0 & 0 & -b \end{array}\right).$$

There are less than three pivot columns if a = b or b = 0.

- (c) $\operatorname{rk} A = 3$ if $a \neq b \neq 0$. $\operatorname{rk} A = 2$ if b = 0 or $a = b \neq 0$. $\operatorname{rk} A = 1$ if b = a = 0.
- (d) For a = b = 2 the row reduced matrix is

$$\left(\begin{array}{ccc} 1 & 1 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{array}\right).$$

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Hence, $x_3 = 0$. If we let $x_2 = a$, then $x_1 = -a$. The general solution takes the form

$$\mathbf{x} = a \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

3. (40pts.) Let
$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$
.

- (a) For what vectors $\mathbf{b} = (b_1, b_2, b_3)^T$ does the linear system $A\mathbf{x} = \mathbf{b}$ have a solution?
- (b) Prove that the column space of A is made up of those vectors $(x,y,z)^T \in \mathbb{R}^3$ that satisfy x+y+z=0.
- (c) Prove that the vectors $(x, y, z)^T \in \mathbb{R}^3$ that satisfy x + y + z = c form a subspace of \mathbb{R}^3 if and only if c = 0.
- (a) The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ -1 & 1 & 0 & b_2 \\ 0 & -1 & 1 & b_3 \end{array}\right).$$

Performing row operations we obtain

$$\left(\begin{array}{cc|cc} 1 & 0 & -1 & b_1 \\ 0 & 1 & -1 & b_1 + b_2 \\ 0 & 0 & 0 & b_1 + b_2 + b_3 \end{array}\right).$$

Hence, the vectors **b** for which $A\mathbf{x} = \mathbf{b}$ has a solution must satisfy $b_1 + b_2 + b_3 = 0$.

- (b) See part (a).
- (c) Suppose that the vectors $\mathbf{x} \in \mathbb{R}^3$, which satisfy x+y+z=c form a subspace. Since $\mathbf{0}$ must be in that subspace, and 0+0+0=0, it follows that c=0.

Now suppose that c = 0. Since 0 + 0 + 0 = 0, $\mathbf{0}$ is in the space. Let \mathbf{x}_1 and \mathbf{x}_2 be in the subspace, and let a and b be real numbers. Consider $a\mathbf{x}_1 + b\mathbf{x}_2$,

$$(ax_1 + bx_2) + (ay_1 + by_2) + (az_1 + bz_2) = a(x_1 + y_1 + z_1) + b(x_2 + y_2 + z_2) = a \cdot 0 + b \cdot 0 = 0.$$

So $a\mathbf{x}_1 + b\mathbf{x}_2$ is also in the space, and therefore this is a subspace of \mathbb{R}^3 .