

## 18.06 Midterm Exam 3, Spring, 2001

Name \_\_\_\_\_

Optional Code \_\_\_\_\_

Recitation Instructor \_\_\_\_\_

Email Address \_\_\_\_\_

Recitation Time \_\_\_\_\_

This midterm is closed book and closed notes. No calculators, laptops, cell phones or other electronic devices may be used during the exam.

There are 3 problems. Good luck.

1. (40pts.) Consider the matrix

$$A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{pmatrix}.$$

- (a) Given that one eigenvalue of  $A$  is  $\lambda = 6$ , find the remaining eigenvalues.
- (b) Find three linearly independent eigenvectors of  $A$ .
- (c) Find an *orthogonal* matrix  $Q$  and a diagonal matrix  $\Lambda$ , so that  $A = Q\Lambda Q^T$ .

2. (20pts.) Consider the system of first order linear ODEs

$$\frac{dx}{dt} = -7x + 2y \quad \frac{dy}{dt} = -6x.$$

Find two independent real-valued solutions  $\begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix}$  and  $\begin{pmatrix} x^{(2)} \\ y^{(2)} \end{pmatrix}$  of this system and hence

find the solution  $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$  which satisfies the initial condition  $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

3. (40pts.) Let  $A_n$  be the  $n \times n$  tridiagonal matrix

$$A_n = \begin{pmatrix} 1 & -a & 0 & 0 & \cdots & 0 \\ -a & 1 & -a & 0 & \cdots & 0 \\ 0 & -a & 1 & -a & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -a & 1 & -a \\ 0 & 0 & \cdots & 0 & -a & 1 \end{pmatrix}.$$

(a) Show for  $n \geq 3$  that

$$\det(A_n) = \det(A_{n-1}) - a^2 \cdot \det(A_{n-2}). \quad (1)$$

(b) Show that eq.(1) can equivalently be written as  $\mathbf{x}_n = B \mathbf{x}_{n-1}$ , where

$$\mathbf{x}_n = \begin{pmatrix} \det(A_n) \\ \det(A_{n-1}) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -a^2 \\ 1 & 0 \end{pmatrix}.$$

(c) For  $a^2 = \frac{3}{16}$ , find an expression for  $\det(A_n)$  for any  $n$ . (*Hint:* One method starts by writing  $B$  in the form  $B = S\Lambda S^{-1}$ , where  $\Lambda$  is a diagonal matrix.)