18.06 **Exam 3 Solutions** **May 3, 2000**

Problem 1 (a) Find the eigenvalues by multiplying each eigenvector by $A^{\mathrm{T}}A$: 64, 4, and 0.
(b) The singular values are the square roots of the nonzero eigenvalues of $A^{\mathrm{T}}A$: 8 and 2.
(c) The SVD is

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 2 (a) False. The 2×2 identity matrix is symmetric, but it has plenty of non-perpendicular eigenvectors, e.g., $(1,0)$ and $(1,1)$.
(b) True, because A is diagonalizable using an orthogonal matrix: $A = Q\Lambda Q^{\mathrm{T}}$. Such a matrix is symmetric: $(Q\Lambda Q^{\mathrm{T}})^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}}$.
(c) False. Same example as in part (a).

Problem 3 There is no such value of $d > 0$. To have positive eigenvalues means that A is positive definite. The upper left determinants are 1, $d - 4$, and $12 - 4d$. These are never all positive.

Problem 4 (a) $\lambda = 1$ is repeated. The number of independent $\lambda = 1$ eigenvectors is given by the dimension of $N(A - I)$, which is two. So A has two independent $\lambda = 1$ eigenvectors, so $\lambda = 1$ must be repeated.
(b) A has three independent eigenvectors, so it is diagonalizable, i.e., similar to

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(c) The matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

has the same eigenvalues and number of independent eigenvalues as A, so is similar to A.
(d) The matrix

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

has the same eigenvalues as A but is missing an eigenvector: the rank of $C - I$ is two, so C has only one independent $\lambda = 1$ eigenvector.