18.06

Solutions to Selected Problems from Quiz #2

Prof. Edelman November 9, 1998

(2)

(a) The columns of A are orthogonal and of norm 2. Hence

$$A = Q \cdot (2I), \text{ where } Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2}A.$$

This means R = 2I.

(b)
$$A^{-1} = (Q(2I))^{-1} = (2I)^{-1}Q^{-1} = \frac{1}{2}I \cdot Q^T = \frac{1}{4}A^T = \frac{1}{4}A \text{ since } A^T = A.$$

(4) A is an $n \times n$ matrix.

(a)
$$R(A) = \mathbb{R}^n \Rightarrow \operatorname{rank}(A) = n \Rightarrow \dim C(A) = n \Rightarrow C(A) = \mathbb{R}^n$$
.

(b)
$$N(A) = \mathbb{R}^n \Rightarrow \text{rank } (A) = 0 \Rightarrow C(A) = Z = \text{zero vector space}.$$

(c)
$$N(A^T) = \mathbb{R}^n \Rightarrow \text{rank } (A) = 0 \Rightarrow C(A) = Z$$

(d) Example 1: A = [0].

Example 2:

$$A = \begin{bmatrix} 0 & 0_m \\ \hline I_k & 0 \end{bmatrix}$$

 I_k is the $k \times k$ identity matrix. 0_m is the $m \times m$ zero matrix.

(e)

$$\left. \begin{array}{l} C(A) \perp R(A) \\ N(A) = (R(A))^{\perp} \end{array} \right\} \Rightarrow C(A) \subseteq N(A)$$

Hence $r + r = \dim C(A) + \dim R(A) \le \dim N(A) + \dim R(A) = n$, i.e. $2r \le m$.

Example 2 above with $k = \left[\frac{n}{2}\right]$, m = n - k shows that $2r \leq n$ is the strongest inequality.

(f) From (e), $r \leq \frac{n}{2} < n \Rightarrow A$ is singular $\Rightarrow \det(A) = 0$.