5 November 1997 **Profs. S. Lee and A. Kirillov**

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Hour Exam II for Course 18.06: Linear Algebra

Recitation Instructor:

Your Name: **SOLUTIONS**

Recitation Time:

Lecturer:

Grading

1. **32**
2. **16**
3. **28**
4. **24**

TOTAL: 100

Do all your work on these pages.
No calculators or notes.
Please work carefully, and check your intermediate results whenever possible.
Point values (total of 100) are marked on the left margin.

[10] **1a.** Give a vector $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ that makes $\underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{s}, \underbrace{\begin{bmatrix} 5 \\ 11 \\ -8 \end{bmatrix}}_{t}, \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{v}$ an orthogonal basis for the vector space $\mathbf{R}^3$.

$v = \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$, `or any nonzero multiple of it.`

`Are the vectors` s `and` t `orthogonal to each other? YES.`

`Next, we need to find a nonzero vector orthogonal to` s `and` t.

`For example, the special solution to` $\begin{bmatrix} 1 & 1 & 2 \\ 5 & 11 & -8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

`is orthogonal to` s `and` t.

`The 3 linearly independent (orthogonal!) vectors` s, t, v `span a 3-dimensional subspace of the vector space` $\mathbf{R}^3$ `(all of` $\mathbf{R}^3$`). The vectors are a basis for` $\mathbf{R}^3$.

[10] **1b.** Given that $\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}}_{B} A = \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}}_{BA}$, find $\det(A)$.

$\det(A) = 27.$

$\underbrace{\det(B)}_{-1} \ \det(A) = \underbrace{\det(BA)}_{-27}.$

[12] **1c.** Can you find a matrix A such that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for the left nullspace of A and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for the nullspace of A?

If 'yes', give a matrix A.
If 'no', briefly explain why the matrix A cannot exist.

`YES. For example:` $A = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$.

`The dimension and number of components for each basis indicates that` A `has` $m = 3$ `rows,` $n = 3$ `columns, and rank` $r = 2$`.`

`The left nullspace indicates that the sum of the rows of` A `is` $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$.

`The nullspace indicates that the sum of the columns of` A `is` $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

`The example chose the first 2 columns of` A `to be linearly independent vectors that are orthogonal to the left nullspace of` A`.`

`Column 3 is chosen so that` $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ `is in the nullspace of` A`; the rank remains` 2`.`

[16] **2.** Find an orthogonal basis for the column space of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 4 & 6 \\ 1 & 4 & 6 \end{bmatrix}$.

Using the Gram-Schmidt process, we obtain: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 2 \\ 2 \end{bmatrix}$.

The orthogonal vectors are denoted as A, B, C:

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$B = \begin{bmatrix} 0 \\ 4 \\ 4 \\ 4 \end{bmatrix} - \left(\tfrac{12}{4}\right) \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{A} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$C = \begin{bmatrix} 0 \\ 0 \\ 6 \\ 6 \end{bmatrix} - \left(\tfrac{12}{4}\right) \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{A} - \left(\tfrac{12}{12}\right) \underbrace{\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{B} = \begin{bmatrix} 0 \\ -4 \\ 2 \\ 2 \end{bmatrix}.$$

3. Let $A = \begin{bmatrix} 1 & -2 & -5 & 1 \\ 2 & -4 & -10 & 3 \end{bmatrix}$.

[16] **3a.** Find the solution to $Ax = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ that is closest to $\begin{bmatrix} 5 \\ 5 \\ 11 \\ 11 \end{bmatrix}$.

$$x = \begin{bmatrix} 7 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Gaussian elimination reveals that the nullspace of A spans a 2-dimensional subspace of $\mathbf{R}^4$.

In particular, the special solutions $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ give a basis for this subspace.

The vector $\begin{bmatrix} 5 \\ 5 \\ 11 \\ 11 \end{bmatrix}$ is closest to $c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, where $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ satisfies

$$\underbrace{\begin{bmatrix} 2 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\begin{bmatrix} 5 & 10 \\ 10 & 26 \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 11 \\ 11 \end{bmatrix}}_{\begin{bmatrix} 15 \\ 36 \end{bmatrix}}.$$

The coefficients are $c_1 = 1$ and $c_2 = 1$.

[12] **3b.** Give an ortho<u>normal</u> basis for the nullspace of $A = \begin{bmatrix} 1 & -2 & -5 & 1 \\ 2 & -4 & -10 & 3 \end{bmatrix}$.

$$\boxed{q_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad q_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}.}$$

The special solutions to $A = \begin{bmatrix} 1 & -2 & -5 & 1 \\ 2 & -4 & -10 & 3 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ give a <u>linearly independent</u> basis for the nullspace of A (see Problem 3a.).

Next, we make an orthogonal basis for the nullspace.

The orthogonal vectors are denoted as A, B:

$$A = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$B = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \left(\tfrac{10}{5}\right) \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{A} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

We make an ortho<u>normal</u> basis by scaling these vectors to have unit length.

$\|A\| = \sqrt{5}$ and $\|B\| = \sqrt{6}$.

So, $q_1 = A/\|A\|$ and $q_2 = B/\|B\|$.

4. Let $A = \begin{bmatrix} 5 & -12 \\ 2 & -5 \end{bmatrix}$.

[8] **4a.** Find the eigenvalues of A.

$$\boxed{\lambda_1 = 1\,,\ \ \lambda_2 = -1\,.}$$

$$\det(A - \lambda I) = \underbrace{(5-\lambda)(-5-\lambda) - (-24)}_{-25+\lambda^2+24} = \lambda^2 - 1 = 0\,.$$

[8] **4b.** Find an eigenvector for each eigenvalue of A.

$$\boxed{x_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \ \ x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.}$$

$$\underbrace{\begin{bmatrix} 4 & -12 \\ 2 & -6 \end{bmatrix}}_{(A-\lambda_1 I)} \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{x_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad \underbrace{\begin{bmatrix} 6 & -12 \\ 2 & -4 \end{bmatrix}}_{(A-\lambda_2 I)} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{x_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

[8] **4c.** Find A^{99}. Recall that $A = \begin{bmatrix} 5 & -12 \\ 2 & -5 \end{bmatrix}$.

$$\boxed{A^{99} = \begin{bmatrix} 5 & -12 \\ 2 & -5 \end{bmatrix} = A\,.}$$

In terms of eigenvalues and eigenvectors,

$$A = \underbrace{\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}}_{S^{-1}} \text{ and } A^{99} = \underbrace{\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1^{99} & 0 \\ 0 & (-1)^{99} \end{bmatrix}}_{\Lambda^{99}} \underbrace{\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}}_{S^{-1}}.$$

A^{99} is the same as A since $\underbrace{\begin{bmatrix} 1^{99} & 0 \\ 0 & (-1)^{99} \end{bmatrix}}_{\Lambda^{99}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda}$.