MIT 18.06 Final Exam, Fall 2022
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Recitation: ______________________________

Problem 1 [5+10 points]:

$Ax = b$ has solutions $x_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, and possibly other solutions, for some (real) matrix A and right-hand side b.

(a) A is an $m \times n$ matrix with rank r. Give as **much true information as possible** about m, n, r. (For example, "$m = 16, r = 0, n \leq 12$" is a possible, but incorrect, answer.)

(b) Give another solution $x_3 =$ ________ (different from x_1 and x_2) for the same equation $Ax = b$. You can do this because you know a nonzero vector ________ in the ________ space of A.

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Problem 2 [10+5 points]:

Robert "Bobby Boy" Boyle (way back in 1662) measured a sequence of m data points $(p_1, v_1), (p_2, v_2), \ldots, (p_m, v_m)$ relating the pressure p of a gas to its volume v. Suppose that he wanted to fit his data to a model of the form

$$V(P) = \alpha + \frac{\beta}{P}$$

and solve for the unknown coefficients α and β that minimize the sum-of-squares error $\sum_k [v_k - V(p_k)]^2$ between the model and the measured data.

(a) Write down a ____$\times$____ system of linear equations (matrix?)(unknowns?) = (right-hand side?) that Bobby could solve to find these best-fit coefficients α and β. You can leave the matrix and right-hand-side as products of terms involving other matrices and/or vectors, but **clearly describe how each term** is constructed from the data $(p_1, v_1), (p_2, v_2), \ldots, (p_m, v_m)$.

(b) Using these best-fit α and β values, the vector $\delta = \begin{pmatrix} v_1 - V(p_1) \\ v_2 - V(p_2) \\ \vdots \\ v_m - V(p_m) \end{pmatrix}$ of discrepancies between the model and the data is an orthogonal projection of the vector ________ onto the ________ space of the matrix ________ .

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Problem 3 [5+10 points]:

Consider the system of differential equations

$$\frac{dx}{dt} = \begin{pmatrix} -1 & 2 \\ & a \end{pmatrix} x$$

with initial condition $x(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

(a) For what value(s) of a will the solution $x(t)$ approach a nonzero constant vector at large t?

(b) Using the value of a from the previous part, write down the exact solution $x(t)$ (at all times, not just for large t).

(blank page for your work if you need it)

Problem 4 [4+4+4+4+4 points]:

The following short-answer questions are answered independently (and refer to unrelated matrices A for each part), requiring little or no computation:

(a) Any solution x of $Ax = b$ is a sum of a vector in the ________ space of A and a vector in the in the ________ space of A.

(b) If $Ax = b$ is solvable for *any* b, then it might be a (**circle one**) 10×3 or 3×10 matrix with rank $r =$ ________. If $Ax = b$ has a *unique* solution x for some b then it might be a (**circle one**) 10×3 or 3×10 matrix with rank $r =$ ________.

(c) Relate the four fundamental subspaces of $A^T A$ to the four fundamental subspaces of a real matrix A: nullspace of $A^T A =$ ________ space of A, left nullspace of $A^T A =$ ________ space of A, column space of $A^T A =$ ________ space of A, row space of $A^T A =$ ________ space of A.

(d) Suppose we solve $A^T A\hat{x} = A^T b$ for $\hat{x}$ given some real A. Then, the orthogonal projection of b into $C(A)$ is the vector ________ and the projection of b onto $N(A^T)$ is the vector ________. (Give formulas in terms of $A, b, \hat{x}$ involving *no matrix inverses*.)

(e) Which of the following matrices **cannot** be singular for **any** real square matrix A (circle **all** answers): $A^T A$, $A^2 + I$, $(A + A^T)^2 + I$, e^{-A}, $A + 10^{100} I$, $3A^T A + 4I$.

(blank page for your work if you need it)

Problem 5 [10+5+5 points]:

Suppose you have a matrix $A = C^{-1}B$ where

$$B = \begin{pmatrix} 1 & & \\ -1 & 2 & \\ 2 & 1 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & & 4 \\ & 2 & 2 \\ 4 & 2 & 2 \end{pmatrix}.$$

The following parts can be **answered independently**.

(a) Compute the **first column** of A^{-1}.

(b) Compute the **trace** of the matrix $A^{-1}B$. (Little calculation is required because $A^{-1}B$ has the same trace, and the same eigenvalues, as ________, since the two matrices are ________!)

(c) One of the eigenvalues of C is $\lambda_1 = 2$. A corresponding eigenvector is $x_1 =$ ________.

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Problem 6 [4+4+4+4+4 points]:

The matrix A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -2$, and $\lambda_3 = 0$, with corresponding eigenvectors $x_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $x_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. Consider the recurrence

$$Ay_{n+1} = y_n - 3y_{n+1},$$

starting with some initial vector y_0.

(a) Give an exact formula for $y_n =$ ________ in terms of A, I, y_0, n. (For example, $y_n = (e^{nA} + 7I)y_0$ is a possible but incorrect answer.)

(b) For a typical initial vector y_0 (e.g. one chosen at random with `randn(3)` in Julia), you should expect y_n for large n to be approximately parallel to the vector ________ and **growing/decaying/oscillating/nearly constant** with n (circle **one**).

(c) Give an example of an initial vector $y_0 =$ ________ for which y_n is **decaying** towards zero with n, and for this y_0 give an *exact* numeric formula (in terms of n) for $y_n =$ ________. (There are many possible answers, but not much calculation should be needed.) Your answer should have no matrices or unknowns, only vectors of numbers or simple arithmetic expressions like 2^n or e^n or $\frac{1}{n^2}$.

(d) The matrix A **can/must/cannot be** Hermitian (circle **one**). Briefly justify your answer.

(e) For $y_0 = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix}$, give a good approximate formula for $y_{100} =$ ________ (numeric vector, no unknowns or matrices).

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Problem 7 [5+8+5 points]:

The real Hermitian (real-symmetric) matrix A has an eigenvalue $\lambda_1 = -\frac{1}{2}$ *(clarification: with multiplicity 1, not a repeated root)* and a corresponding eigenvector $x_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$, and its other eigenvalues are all equal to 1.

(a) Give *one* example of an eigenvector of A for $\lambda_2 = 1$.

(b) The orthogonal projection of $b = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ onto the span S of x_1 is ________ and the projection of b onto the orthogonal complement $S^{\perp}$ is ________.

(c) With the help of the previous part, an *exact* formula for $A^n \begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} =$ ________ (in terms of n and explicit numerical vectors, no matrices or unknowns).

(blank page for your work if you need it)

Problem 8 [5+8+5 points]:

Suppose that Q is a 4×3 real matrix with orthonormal columns q_1, q_2, q_3.

(a) Starting from a real vector v (not in the column space of Q), give a formula for the fourth orthonormal vector q_4 that would be produced by Gram–Schmidt on q_1, q_2, q_3, v.

(b) Describe $N(Q)$, $N(Q^T)$, $N(Q^TQ)$, and $N(QQ^T)$: give the dimension and a basis for each (in terms of q_1, q_2, q_3, q_4 as needed).

(c) Suppose $b = q_1 + 2q_2 + 3q_3 + 4q_4$. Give the least-squares solution $\hat{x} =$ ________ minimizing $\|b - Qx\|$.

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