MIT 18.06 Exam 3 **Solutions**, Fall 2022
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Problem 1 [10+(4+4)+5+10 points]:

Two of the eigenvectors of the **real** matrix A are $x_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$ with corresponding eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2 + i$.

(a) Another eigenvalue of A is $\lambda_3 = $ ____, and A is a _ $\times$ _ matrix equal to $A = $ ____. You can leave your answer for A as a product of matrices and/or matrix inverses without simplifying.

(b) $\det A = $ ____ and trace $A = $ ____.

(c) $\det(A - \lambda I) = $ ____ (simplify to a polynomial in λ). (Time-saving hint: You can do this without calculating A explicitly!)

(d) Give *all* of the eigenvalues, and corresponding eigenvectors, of $(A^2 - 2I)e^{(A^{-1})}$. You can leave your eigenvalues as **non-simplified** arithmetic expressions.

Solutions:

(a) A is real, so the eigenvalues and eigenvectors must come in complex-conjugate pairs. So, $\boxed{\lambda_3 = \overline{\lambda_2} = 2 - i}$ and a corresponding eigenvector is $x_3 = \overline{x_2} = \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$. Since the eigenvectors have 3 components, A must be a $\boxed{3 \times 3}$ matrix, andd we can compute it using the diagonalization (since we have a basis of 3 independent eigenvectors for 3 distinct eigenvalues):

$$\boxed{A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & -i \\ 1 & 1 & 1 \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} 1 & & \\ & 2+i & \\ & & 2-i \end{pmatrix}}_{\Lambda} X^{-1}}.$$

(b) $\det A = \lambda_1\lambda_2\lambda_3 = 1\times|2+i|^2 = 1\times(2^2+1^2) = \boxed{5}$. $\operatorname{trace} A = \lambda_1+\lambda_2+\lambda_3 = \lambda_1 + 2\operatorname{Re}\lambda_2 = \boxed{5}$.

(c) This is the characteristic polynomial, and it is the same as the characteristic polynomial of Λ (since similar matrices have the same characteristic polynomial), so it must be $\det(A-\lambda I) = \det(\Lambda-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda)$: its roots are $\lambda_1, \lambda_2, \lambda_3$, and the leading term must be $-\lambda^3$ as can be seen from the diagonal-matrix determinant $\det(\Lambda-\lambda I)$. This simplifies further to:

$$(\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda) = (1-\lambda)(2+i-\lambda)(2-i-\lambda) = (1-\lambda)(\lambda^2-4\lambda+5) = \boxed{-\lambda^3+5\lambda^2-9\lambda+5}.$$

Note that this is a purely real polynomial as expected (since A is real), despite the presence of complex roots.

A common error was to get the sign wrong. Remember that $\det(A-\lambda I) \neq \det(\lambda I - A) = \prod_k(\lambda-\lambda_k)$ unless A is an $m\times m$ matrix where m is even, since $\det(-A) = (-1)^m \det(A)$. Of course, $\det(\lambda I - A)$ has the same roots, but it is not quite what was asked for.

Some people gave an answer of "0", which of course would be correct for $\det(A-\lambda_k I)$ with $k=1,2,3$, i.e. if λ were one of the eigenvalues. But λ here was *not* specified to be one of the eigenvalues λ_k in this problem.

Several people used the formula $\det(A-\lambda I) = \lambda^2 - \operatorname{trace}(A)\lambda + \det(A)$, but this **only** applies to 2×2 matrices! (It only has 2 roots!) An even more egregious error is to write "$\det(A-\lambda I) = \det(A) - \lambda\det(I)$" or similar—the determinant is *not* a linear function (it is only linear in *individual rows or columns*).

(d) This matrix has the **same eigenvectors** $\boxed{x_1, x_2, x_3}$ and corresponding eigenvalues $\boxed{(\lambda_k^2-2)e^{\lambda_k^{-1}}}$ for $k=1,2,3$, respectively. More explicitly, $\lambda_1, \lambda_2, \lambda_3 = -e, (1+4i)e^{\frac{1}{2+i}}, (1-4i)e^{\frac{1}{2-i}}$.

Problem 2 [11+11+11 points]:

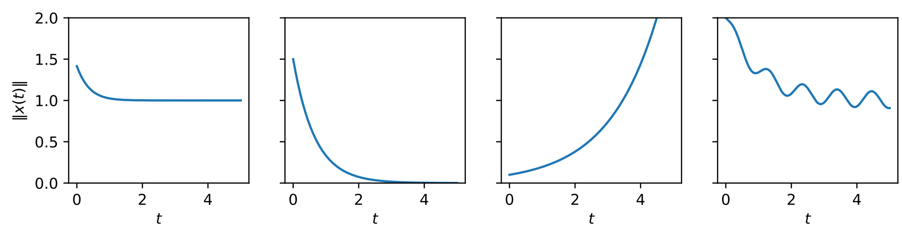
Consider the differential equation

$$\frac{dx}{dt} = -B^T B x, \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 3 \\ 4 & 0 & 4 \\ 5 & 1 & 5 \end{pmatrix}.$$

(a) $x(t) =$ (constant vector) is a possible solution of this ODE for what vector(s) x? (Describe *all* possible answers. Look carefully at B!)

(b) Which of the following looks like a possible plot of $\|x(t)\|$ versus t for some initial $x(0)$? Circle **all possibilities**. (Note: all vertical axes are identical.)

You know this because the eigenvalues of ____ must be ____.



(c) For $x(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, give a **good approximation** for $x(1000) \approx$ ____.
(Give a specific numerical vector, no unknowns.)

Solutions:

(a) A constant vector is a solution when $\frac{dx}{dt} = -B^T B x = 0$, i.e. if and only if $x \in N(B^T B)$. But (from exam-2 material), we know $N(B^T B) = N(B)$, and looking carefully at B the nullspace should be obvious: its first and third columns are identical, while the middle column is clearly independent, so $N(B) = \boxed{\text{span}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$ are the possible constant-x solutions, corresponding to multiples of the eigenvector $x_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ for $\lambda_1 = 0$ of $B^T B$.

(b) We know that the eigenvalues of $\boxed{-B^T B}$ must be $\boxed{\leq 0}$ because **any** matrix of this form is negative semidefinite. Furthermore, we know from part (a) that there is **one** $\lambda = 0$ eigenvalue, so the eigenvalue must be $\boxed{\lambda_1 = 0, \lambda_2 < 0, \lambda_3 < 0}$. This means that there are only **exponentially decaying and/or constant solutions** $x(t)$ or any superposition thereof.

Hence the two possibilities are **the left two graphs**: either decaying to a nonzero constant or decaying to zero (if the x_1 component happens to be zero). You *cannot* have growing or oscillating solutions to this ODE, as in the right two graphs.

(From Julia, the other two eigenvalues turn out to be $\lambda_2 \approx -111.493$ and $\lambda_3 \approx -1.5068$. In principle, you could find these analytically by solving a quadratic equation, since you already know one of the roots of the characteristic polynomial, but I *don't* expect you to carry out this calculation on the exam!)

(c) In general, we expect the solutions to look like a superposition of the eigenvectors of $-B^T B$:

$$x(t) = c_1 \cancelto{1}{e^{\lambda_1 t}} x_1 + \underbrace{c_2 e^{\lambda_2 t} x_2 + c_3 e^{\lambda_3 t} x_3}_{\text{exponentially decaying}} \approx c_1 x_1 \text{ for large } t,$$

but the x_2 and x_3 terms are exponentially decaying since the corresponding eigenvalues *must* be < 0 from above. Hence, for a large time $t = 1000$, unless we get *very* unlucky and λ_2, λ_3 are *very* small (which turns out not to be the case here), $x(t)$ will be dominated by the first term $c_1 x_1$.

How do we find the coefficient c_1? For a general eigenproblem, we'd need to solve $Xc = x(0)$, which would require us to calculate x_2 and x_3 and then go through a laborious (for humans) Gaussian-elimination process. But $-B^T B$ is **Hermitian** (real-symmetric) and so x_1 must be **orthogonal** to the other two eigenvectors (for $\lambda_2, \lambda_3 \neq \lambda_1 = 0$). Hence, we can find the coefficient simply by taking a **dot product**, i.e. an **orthogonal projection** onto x_1:

$$x(t) \approx c_1 x_1 = x_1 \frac{x_1^T x(0)}{x_1^T x_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \frac{\cancelto{-2}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}}{2} = \boxed{\begin{pmatrix} -1 \\ 0 \\ +1 \end{pmatrix}}.$$

Problem 3 [10+8+8+8 points]:

Suppose that the sequence of vectors $y_0, y_1, y_2, \ldots \in \mathbb{R}^m$ satisfies the recurrence

$$\frac{y_n - y_{n-1}}{h} = A\left(\frac{y_{n-1} + y_n}{2}\right)$$

for some real $h > 0$ and some $m \times m$ matrix A.

(a) Write $y_n = (\underline{\quad\quad})y_{n-1} = (\underline{\quad\quad})y_0$, where you fill in the blanks with **some matrices** written in terms of A, I (the $m \times m$ identity), h, and n.

(b) If $y_0 = x_k$ where x_k is an **eigenvector** of A with eigenvalue λ_k, give a *much simpler* formula $y_n = \underline{\quad\quad}$ in terms of x_k, λ_k, h, n.

(c) The solutions y_n **must** be **decaying to zero** as $n \to \infty$ if A is (**circle all** that apply): *real, Hermitian, positive-definite, positive-semidefinite, negative-definite, negative-semidefinite.* **Justify** your answer (briefly!).

(d) If $A = iB$ where B is **Hermitian** and **invertible**, then the solutions y_n for $y_0 \neq 0$ must be (**circle one**): growing, decaying to zero, approaching a nonzero constant, oscillating. **Justify** your answer (briefly!).

Solutions:

Side commentary (not relevant to 18.06): this problem is actually motivated by "Crank–Nicolson" schemes for discretizing $\frac{dy}{dt} = Ay$, with $y_n = y(ht)$ for a discrete "timestep" h, and where the time derivative is approximated by a centered "finite difference." Negative-definite and semi-definite A then arise in "parabolic" or "diffusion" type equations (with decaying solutions as in part c), while "anti-Hermitian" $A = -A^H$ arise in "hyperbolic" or "wave" equations (with oscillating solutions as in part d).

(a) Simply moving all of the y_n terms to the left and all the y_{n-1} terms to the right gives $\frac{y_n}{h} - \frac{A}{2}y_n = \frac{y_{n-1}}{h} + \frac{A}{2}y_{n-1}$, which can be rewritten as

$$\left(\frac{I}{h} - \frac{A}{2}\right) y_n = \left(\frac{I}{h} + \frac{A}{2}\right) y_{n-1}$$

$$\Longrightarrow \boxed{y_n = \underbrace{\left(\frac{I}{h} - \frac{A}{2}\right)^{-1}\left(\frac{I}{h} + \frac{A}{2}\right)}_{C} y_{n-1} = \underbrace{\left(I - \frac{h}{2}A\right)^{-1}\left(I + \frac{h}{2}A\right)}_{C} y_{n-1}}$$

(where the latter version is obtained by multiplying and dividing by h. So, $y_1 = Cy_0$, $y_2 = Cy_1 = C^2 y_0$, and so on, giving us

$$\boxed{y_n = C^n y_0}$$

where C is the matrix defined above.

Note that for this recurrence to exist, we must have $I - \frac{h}{2}A$ invertible, so that assumption is arguably implicit in the problem.

Note also that the matrices $(I - \frac{h}{2}A)^{-1}$ and $(I + \frac{h}{2}A)$ in fact commute, so there are lots of additional correct ways to write this solution.

(b) If $Ax_k = \lambda_k x_k$, then x_k is **also** an eigenvector of C with eigenvalue $\mu_k = \frac{1+\frac{h}{2}\lambda_k}{1-\frac{h}{2}\lambda_k} = \frac{\frac{1}{h}+\frac{1}{2}\lambda_k}{\frac{1}{h}-\frac{1}{2}\lambda_k} = \frac{\frac{2}{h}+\lambda_k}{\frac{2}{h}-\lambda_k}$, hence

$$y_n = C^n x_k = \mu_k^n x_k = \boxed{\left(\frac{1+\frac{h}{2}\lambda_k}{1-\frac{h}{2}\lambda_k}\right)^n x_k = \left(\frac{\frac{1}{h}+\frac{1}{2}\lambda_k}{\frac{1}{h}-\frac{1}{2}\lambda_k}\right)^n x_k = \left(\frac{\frac{2}{h}+\lambda_k}{\frac{2}{h}-\lambda_k}\right)^n x_k},$$

where again we can write the answer in several equivalent forms.

(c) The **key to both parts (c) and (d)** is to understand that we need to **analyze the eigenvalues** μ_k **of** C, and in particular we need to know something about the *magnitudes* $|\mu_k|$ in order to know what happens to matrix *powers* $C^n x_k$.

To be decaying, we must have all eigenvalues μ_k of C satisfy $|\mu_k| < 1$. Under which of the listed conditions is this guaranteed?

It is clearly not sufficient for A to be real, or even Hermitian. If λ_k could be any real number (as for a Hermitian matrix), then it could be a positive real number, and by inspection $|\mu_k| > 1$ if $\lambda_k > 0$ (since the numerator is bigger than the denominator). By the same token, we cannot have A positive definite or semidefinite. We also cannot have A negative semidefinite, since that would allow $\lambda_k = 0$ eigenvalues, giving $\mu_k = 1$ (a steady state, not decaying, solution).

However, the last possibility works: if A is negative definite, then its eigenvalues are all $\lambda_k < 0$, which by inspection gives $|\mu_k| < 1$ since the numerator is smaller in magnitude than the denominator (numerator = subtraction, denominator = addition).

Of course, if A is negative-definite then it must also be negative-semidefinite and Hermitian, but the latter are not sufficient by *themselves*.

(d) If $A = iB$ where B is Hermitian (real eigenvalues), then all of the eigenvalues of A are **purely imaginary** ($i \times$real). If B is invertible, then none of the eigenvalues are zero. So $A's$ eigenvalues λ_k have the form $\lambda_k = ib_k$ where b_k is a real number $\neq 0$.

What does this tell us about the eigenvalues μ_k of C? Well, we have

$$\mu_k = \frac{\frac{2}{h}+ib_k}{\frac{2}{h}-ib_k} \implies |\mu_k| = \frac{\left|\frac{2}{h}+ib_k\right|}{\left|\frac{2}{h}-ib_k\right|} = 1,$$

since the numerator and denominator are **complex conjugates** (which have the same magnitude). Furthermore, since $b_k \neq 0$, we have $\mu_k \neq 1$. Therefore, since the eigenvalues of B are complex numbers with **unit magnitude** but $\neq 1$, we must have **oscillating** solutions y_n for any $y_0 \neq 0$.