MIT 18.06 Exam 2, Fall 2022
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(*printed*)

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Recitation: ______________________________

Problem 1 [(5+5)+10 points]:

These two parts are **answered independently:**

(a) Consider the 2d "plane" S spanned by

$$a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; a_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(i) Give an **orthonormal basis** for S.

(ii) Find the **closest point** in S to the (column vector) $y = [-2, 4, -6, 8]$.

(b) Suppose that we have 100 measurements (p_k, v_k) of the volume v of a gas vs. its pressure p, and we want to fit it to a function of the form $v(p) = \frac{c_1}{p} + c_2$ for unknown constants c_1, c_2. Write down the 2×2 **system of equations** you would solve to find c_1, c_2 in order to minimize the sum of the squared errors $\sum_k [v(p_k) - v_k]^2$. You can write your answer (left- and right-hand sides) as products of matrices and/or vectors, as long as you specify what each term is (in terms of the unknowns c_1, c_2 and/or the data $p_1, \ldots, p_{100}$ and $v_1, \ldots, v_{100}$).

(blank page for your work if you need it)

Problem 2 [4+4+4+4+4+4 points]:

These parts can be **answered independently:**

(a) The matrix $\frac{a_1 a_1^T}{a_1^T a_1} + \frac{a_2 a_2^T}{a_2^T a_2}$ is the projection matrix onto the span of $a_1, a_2 \in \mathbb{R}^m$ if a_1 and a_2 are **(circle all true answers)**: *independent, orthogonal, parallel, orthonormal, singular, length-1.*

(b) If $\hat{x}$ is the least-square solution minimizing $\|Ax - b\|$ over x, then $A\hat{x} - b$ must lie in **which fundamental subspace** of A?

(c) A, B are 10×3 matrices, and $b \in \mathbb{R}^{10}$. If we want to find the vector $\hat{y} \in \mathbb{R}^3$ for which $A\hat{y} - b \in C(B)^\perp$, then $\hat{y}$ satisfies the 3×3 **system of equations** ______ (in terms of $A, B, b, \hat{y}$).

(d) A, B are matrices with $C(A) = C(B)$, and we have solved $A^T A\hat{x} = A^T b$ for $\hat{x}$ and $B^T B\hat{y} = B^T b$ for $\hat{y}$. **Circle statements (if any) that *must* be true:** $\hat{x} = \hat{y}$, $A\hat{x} = B\hat{y}$, and/or $\hat{x}^T b = \hat{y}^T b$.

(e) Q is a 5×3 matrix with orthonormal columns. Circle which **must** be true: $\|Qx\| = \|x\|$ for $x \in \mathbb{R}^3$, $\|Q^T y\| = \|y\|$ for $y \in \mathbb{R}^5$.

(f) If A is a 3×3 matrix with $\det(A) = 3$, then $\det[A^T A^{-1}] + \det(2A) =$ ____.

(blank page for your work if you need it)

Problem 3 [(3+3+3)+5 points]:

These two parts are **answered independently:**

(a) If A is a 10×3 matrix has an SVD $U\Sigma V^T$ with $\Sigma = \begin{pmatrix} 100 & & \\ & 10 & \\ & & 1 \end{pmatrix}$, then

(i) U is a ____ $\times$ ____ matrix, V is a ____ $\times$ ____ matrix, and A has rank ____.

(ii) The projection matrix onto $C(A)$ is _______ and the projection onto $C(A^T)$ is _______ (**simplest answers** in terms of U, Σ, V, I).

(iii) A good rank-2 approximation for A is _______ (in terms of U, V)

(b) If$f(x) = (x^T y)^2$ for $x, y \in \mathbb{R}^n$, then give a formula for ∇f (in terms of y and/or x).

(blank page for your work if you need it)