**Final Exam Review Session #1**
December 12, 2019

1. Compute the $LDU$ factorization of the following matrix:

$$X = \begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & -1 \\ -3 & 4 & 2 \end{bmatrix}.$$

**Solution:**

$$\begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & -1 \\ -3 & 4 & 2 \end{bmatrix} \xrightarrow{r_2 \mapsto 2r_1 + r_2} \begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & -1 \\ 0 & 4 & -4 \end{bmatrix} \xrightarrow{r_3 \mapsto -3r_1 + r_3} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 4 & -4 \end{bmatrix} \xrightarrow{r_3 \mapsto -4r_2 + r_3} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -16 \end{bmatrix},$$

so

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & -1 \\ -3 & 4 & 2 \end{bmatrix}.$$

(Recall that matrix multiplication on the left corresponds to taking linear combinations of the rows!) That is, letting $Y = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -16 \end{bmatrix}$, we have that $Y = E_{32}^{(-4)} E_{31}^{(-3)} E_{21}^{(2)} X$. However, $Y = DU$ where $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -16 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, so letting $L = (E_{32}^{(-4)} E_{31}^{(-3)} E_{21}^{(2)})^{-1} = E_{21}^{(-2)} E_{31}^{(3)} E_{32}^{(4)}$ it follows that $X = LDU$. Explicitly,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}.$$

2. (a) Find a basis for the null space of the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & 3 \\ 2 & -6 & 1 & -3 \\ 3 & 0 & 6 & 9 \end{bmatrix}.$$

**Solution:** We perform Gauss-Jordan elimination to find the reduced row echelon form of $A$:

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & 3 \\ 2 & -6 & 1 & -3 \\ 3 & 0 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & 3 \\ 2 & -6 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & 3 \\ 0 & -6 & -3 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Letting $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ be a general element of the null space, we find that

$$x_1 = -2x_3 - 3x_4$$

and

$$x_2 = -\frac{1}{2}x_3 - \frac{3}{2}x_4,$$

so

$$\boldsymbol{x} = x_3 \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} - x_4 \begin{bmatrix} -3 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}.$$

Therefore $\left\{ \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $N(A)$.

(b) Let

$$\boldsymbol{b} = A \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Find the general solution $\boldsymbol{v}_{general}$ to $A\boldsymbol{v} = \boldsymbol{b}$.

**Solution:** We have that the general solution $\boldsymbol{v}_{general} = \boldsymbol{v}_{particular} + \boldsymbol{w}_{general}$, where $\boldsymbol{v}_{particular}$ is a specific solution to $A\boldsymbol{v} = \boldsymbol{b}$ and $\boldsymbol{w}_{general}$ is the general solution to $A\boldsymbol{w} = 0$ (i.e. a general element of $N(A)$). Because $\boldsymbol{b} = A \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, it follows that $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution. Therefore, using our computation from (a) we find that

$$\boldsymbol{v}_{general} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -3 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix},$$

where $\alpha, \beta \in \mathbb{R}$.

3. (a) Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, and $\boldsymbol{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, and let $B = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}$. Compute the QR factorization of $B$.

**Solution:** We perform the Gram-Schmidt process:

$$\boldsymbol{v}_1 \to \boldsymbol{w}_1 = \boldsymbol{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}$$

$$\boldsymbol{v}_2 \to \boldsymbol{w}_2 = \boldsymbol{v}_2 - (\boldsymbol{v}_2 \cdot \boldsymbol{q}_1)\boldsymbol{q}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + \frac{4}{5} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{2}{5} \\ -1 \end{bmatrix}$$

$$\boldsymbol{w}_2 \to \boldsymbol{q}_2 = \frac{\sqrt{5}}{3} \begin{bmatrix} \frac{4}{5} \\ \frac{2}{5} \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3\sqrt{5}} \\ \frac{2}{3\sqrt{5}} \\ -\frac{\sqrt{5}}{3} \end{bmatrix} = \frac{1}{3\sqrt{5}} \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}$$

$$\boldsymbol{v}_3 \to \boldsymbol{w}_3 = \boldsymbol{v}_3 - (\boldsymbol{v}_3 \cdot \boldsymbol{q}_1)\boldsymbol{q}_1 - (\boldsymbol{v}_3 \cdot \boldsymbol{q}_2)\boldsymbol{q}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{5}{9} \\ \frac{2}{9} \\ -\frac{5}{9} \end{bmatrix} = \frac{2}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$w_3 \to q_3 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

We therefore set

$$Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} & \frac{1}{3} \\ 0 & -\frac{\sqrt{5}}{3} & \frac{2}{3} \end{bmatrix}.$$

The preceding computations tell us that

$$Q = BD_1^{(1/\sqrt{5})} E_{12}^{(4/\sqrt{5})} D_2^{\sqrt{5}/3} E_{13}^{(\sqrt{5})} E_{23}^{(\sqrt{5}/3)} D_3^{(3/2)},$$

so letting $R = (D_1^{(1/\sqrt{5})} E_{12}^{(4/\sqrt{5})} D_2^{\sqrt{5}/3} E_{13}^{(\sqrt{5})} E_{23}^{(\sqrt{5}/3)} D_3^{(3/2)})^{-1}$, we have that $B = QR$. Explicitly,

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & -\sqrt{5} \\ 0 & 1 & -\frac{\sqrt{5}}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{4}{\sqrt{5}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{5} & -\frac{4}{\sqrt{5}} & -\sqrt{5} \\ 0 & \frac{3}{\sqrt{5}} & -\frac{\sqrt{5}}{3} \\ 0 & 0 & \frac{2}{3} \end{bmatrix}.$$

(b) Let $U$ be the subspace of $\mathbb{R}^3$ spanned by $v_1$ and $v_2$. Compute $P_U$, the projection onto $U$.

**Solution:** Let $A = \begin{bmatrix} q_1 & q_2 \end{bmatrix}$. Then, by the orthonormality of $\{q_1, q_2\}$, we see that

$$P_U = A(A^T A)^{-1} A^T = AA^T = q_1 q_1^T + q_2 q_2^T$$

$$= \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{16}{45} & \frac{8}{45} & -\frac{4}{9} \\ \frac{8}{45} & \frac{4}{45} & -\frac{2}{9} \\ -\frac{4}{9} & -\frac{2}{9} & \frac{5}{9} \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & -\frac{2}{9} & -\frac{4}{9} \\ -\frac{2}{9} & \frac{8}{9} & -\frac{2}{9} \\ -\frac{4}{9} & -\frac{2}{9} & \frac{5}{9} \end{bmatrix}.$$

(c) Let $W$ be the orthogonal complement of $U$. What is a basis for $W$? Compute $P_W$, the projection onto $W$.

**Solution:** We see that $\{q_3\}$ is a basis for $W$. Then,

$$P_W = q_3 q_3^T = \begin{bmatrix} \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \end{bmatrix}.$$

Alternatively, we could have observed that $P_W$ must equal $I - P_U$ because $U$ and $W$ are orthocomplements of each other.

4. Let

$$T = \begin{bmatrix} -2 & 3 & -4 \\ 1 & -2 & 3 \\ 3 & -4 & 4 \end{bmatrix}.$$

Compute $\det(T)$ by row operations, cofactor expansion, and the big formula.

**Solution:** We first compute the determinant of $T$ by row operations:

$$\begin{bmatrix} -2 & 3 & -4 \\ 1 & -2 & 3 \\ 3 & -4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 3 & -4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 2 \\ 0 & 2 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$$

The determinant of $T$ is therefore $(-1)$ (for the row swap done first) multiplied by $1(-1)(-1) = 1$. Thus $\det(T) = -1$.

We next compute the determinant by cofactor expansion along the second row:

$$\det(T) = (-1)^{2+1}(1)(12-16) + (-1)^{2+2}(-2)(-8+12) + (-1)^{2+3}(3)(8-9) = 4 - 8 + 3 = -1.$$

Finally, we compute the determinant by using the big formula:

$$\begin{aligned}\det(T) &= (-1)^{\operatorname{sgn} 1}(-2)(-2)(4) + (-1)^{\operatorname{sgn}(132)}(3)(3)(3) + (-1)^{\operatorname{sgn}(123)}(1)(-4)(-4) \\ &+ (-1)^{\operatorname{sgn}(12)}(1)(3)(4) + (-1)^{\operatorname{sgn}(23)}(-2)(-4)(3) + (-1)^{\operatorname{sgn}(13)}(3)(-2)(-4) \\ &= 16 + 27 + 16 - 12 - 24 - 24 = -1.\end{aligned}$$

Note that here, the determinant was the sum of the products of all the diagonals of $T$ (reversing the signs for diagonals of the opposite direction), but this does not work for matrices of higher dimensions (for one, there are fewer diagonals than permutations in general).