Recitation 9. November 12

Focus: Singular Value Decomposition.
Recall that for a matrix A the **Singular Value Decomposition** (SVD) is an expression $A = U\Sigma V^T$ where U, V are orthogonal matrices and Σ is diagonal.
The **Singular Values** denoted σ_i are the diagonal entries of Σ.
The **Pseudo-inverse** of A is given in terms of the SVD by $A^+ = V\Sigma^+ U^T$ where Σ^+ has diagonal entries $\frac{1}{\sigma_i}$.
A^+A and AA^+ are the projections onto $C(A^T)$ and $C(A)$ respectively.

1. Consider the matrix

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

- Compute the Singular Value Decomposition of A.
- Compute the Psuedo-inverse A^+. Then compute the inverse A^{-1} by another method. How do they compare?

Solution:

2. 1. Find the maximum of the function

$$\frac{3x_1^2 + 2x_1x_2 + 3x_2^2}{x_1^2 + x_2^2}$$

by expressing it in the form $\frac{x^TSx}{x^Tx}$ for a symmetric matrix S and using the relation of this expression to the eigenvalues of S. For what values of (x_1, x_2) is the maximum achieved?

2. Find the minimum of the function

$$\sqrt{\frac{(x_1 + 4x_2)^2}{x_1^2 + x_2^2}}$$

by expressing it in the form $\frac{\|Ax\|}{\|x\|}$ and using the relation of this expression to the singular values of A.

Solution:

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

   1. Compute its singular value decomposition
   2. Use this to find the closest vector to $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ in the column space of A and in the column space of A^T. How else could you compute these vectors? Do the other methods agree?

Solution:

4.
   1. If $A = QR$ is a Gram-Schmidt Orthogonalization of A (i.e. Q is an orthogonal matrix), how does the SVD of A relate to the SVD of R?
   2. If $A = U\Sigma V^T$ is a SVD of a matrix A, and Q_1, Q_2 are two orthogonal matrices, how do the singular values σ_i of $Q_1 A Q_2^{-1}$ relate to those of A?

Solution: